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# Strongly interacting $W$'s and $Z$'s and the elastic unitarity constraint

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By employing the dictum that axiomatic principles are devoid of predictive power, we find that the elastic unitarity constraint, applied to strong $W_L W_L$ scattering, does not alter the assumed spectrum of intermediate states In other words, if one integrates out the heavy degrees of freedom in an effective theory, then imposing elastic unitarity on the resultant low-energy theory effectively integrates back in the original heavy degrees of freedom, but otherwise leaves the spectrum unchanged. As a case study, we consider intermediate states involving a heavy Higgs and heavy fermions of a hypothetical fourth generation doublet In contrast to recent studies, we find no p-wave resonance.

## 1. Introduction

Long ago, it was of interest to determine whether a strongly interacting linear sigma model (LSM), in all its apparent simplicity, could prove capable of generating the complex hadronic spectrum [1]. In recent times, the familiar isomorphism between the pions and the longitudinal modes of the standard model gauge bosons, made precise through the equivalence theorem, has led to a resurgence of interest in the strongly interacting LSM [2–4]. Motivation rests on the understanding that $W_L W_L$ scattering provides a unique probe of the mechanism responsible for electroweak symmetry breaking [5]. Evidently, triviality bounds [6] provide at best a narrow window within which a strongly interacting Higgs sector could exist. Nevertheless, a strongly interacting Higgs sector remains an interesting possibility. The onset of strong interactions is, of course, signaled by the breakdown of perturbative unitarity. Thus, there have been many attempt to make meaningful statements about the electroweak symmetry breaking sector by imposing elastic unitarity as a constraint, an approach known as unitarization. It is the purpose of this letter to show that if unitarization is carried out properly, nothing is learned about the physical spectrum of the symmetry breaking sector.

Unitarization of a scattering amplitude requires trading crossing symmetry for elastic unitarity, in a non-unique way. Beyond requiring that our unitary approximation have good analyticity properties, and be consistent to a certain degree with chiral symmetry and crossing, we impose an additional constraint: In sync with current lore [7], we ensure that unitarity per se yields no predictive power, a point of view clearly orthogonal to $S$-matrix theory (in the bootstrap sense). We consider the derivation of the well-known KSRF relation [8] as a heuristic tool to make clear exactly what we mean by this constraint.

In order to stress the generality of our statements, we make use of the effective lagrangian approach. As will be made clear, this also allows us to maintain a modified power-counting scheme by decomposing the scattering amplitude according to its crossing properties. As a case study, we saturate the undermined constants of chiral perturbation theory with contributions from intermediate states involving a heavy Higgs and heavy degenerate

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fermions of a hypothetical fourth generation doublet. As expected, the only nearby pole of the full amplitude is seen to be the physical Higgs pole.

In refs. [3,4] the method of Padé approximants [#1] is use to show that, for a large fermion mass, it is possible to dynamically generate a p-wave resonance. The $S$ parameter bound can then serve to exclude a heavy fourth generation of fermions [4]. We will argue that the p-wave resonance generated by the Padé method is unphysical, and owes its existence to the incorrect presumption that elastic unitarity should be imposed for the purpose of making predictions.

The plan of the remainder of the letter is as follows. In section 2 we obtain the chiral perturbation series to order $E^4$ by way of a chiral Lagrangian. In section 3 we present our unitarization scheme. We then present our case study in section 4, and contrast our results with other recent work. Finally, in section 5 we present a summary and conclusion.

## 2. Effective Lagrangian

Exploitation of the model-independent low-energy structure of the theory is essential to our approach. Assuming a custodial SU(2) symmetry, the most general effective Lagrangian including terms with four derivatives is given by [#2]

$$\mathscr{L}=\tfrac{1}{4}v^2\mathrm{Tr}(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)+\frac{C_1}{16\pi^2}\mathrm{Tr}(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)\mathrm{Tr}(\partial_\nu\Sigma\partial^\nu\Sigma^\dagger)+\frac{C_2}{16\pi^2}\mathrm{Tr}(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger)\mathrm{Tr}(\partial^\mu\Sigma\partial^\nu\Sigma^\dagger)\,. \tag{2.1}$$

The Goldstone boson fields ($w^+$, $w^-$, and $z$) are contained within the field variable $\Sigma=\exp[\mathrm{i}(\boldsymbol{\tau}\cdot\boldsymbol{\nu})/v]$. $C_1$ and $C_2$ are undetermined constants which characterize the underlying theory at low energies. In general, there are contributions to $C_1$ and $C_2$ from all heavy degrees of freedom, as well as continuum contributions arising from Goldstone boson loops. To order $s^2$, the relevant partial wave amplitudes of definite custodial isospin obtained from this lagrangian are given by

$$a_0(s)\equiv a_{00}(s)$$
$$=\alpha_0 s\left\{1-\frac{\alpha_0 s}{\pi}\left[\log\left(\frac{-s}{\mu^2}\right)-6(2C_1+C_2)\right]-\frac{\alpha_0 s}{\pi}\left[\tfrac{7}{18}\log\left(\frac{s}{\mu^2}\right)-\tfrac{11}{108}-\tfrac{2}{3}(4C_1+5C_2)\right]\right\}, \tag{2.2a}$$

$$a_1(s)\equiv a_{11}(s)$$
$$=\alpha_1 s\left\{1-\frac{\alpha_1 s}{\pi}\left[\log\left(\frac{-s}{\mu^2}\right)-12C_2\right]+\frac{\alpha_1 s}{\pi}\left[\log\left(\frac{s}{\mu^2}\right)-\tfrac{1}{3}+12(C_2-4C_1)\right]\right\}, \tag{2.2b}$$

$$a_2(s)\equiv a_{20}(s)$$
$$=\alpha_2 s\left\{1-\frac{\alpha_2 s}{\pi}\left[\log\left(\frac{-s}{\mu^2}\right)-12C_2\right]-\frac{\alpha_2 s}{\pi}\left[\tfrac{11}{9}\log\left(\frac{s}{\mu^2}\right)-\tfrac{25}{54}-\tfrac{4}{3}(8C_1+7C_2)\right]\right\}, \tag{2.2c}$$

where $\alpha_0\equiv 1/16\pi v^2$, $\alpha_1\equiv 1/96\pi v^2$, and $\alpha_2\equiv -1/32\pi v^2$. EAch curly bracket consists of three terms, corresponding to the low-energy theorem, and the $\mathrm{O}(s^2)$ contributions in the direct- and the crossed-channel respectively. *Note that we have been careful to preserve the crossing properties of the undetermined coefficients* [10]. This decomposition will prove an important ingredient in what follows.

[#1] Although Veltman and Veltman make use of an "effective range" approximation, their amplitude is equivalent to the [1,1] Padé approximant.

[#2] The coefficients, normalized in this way, are of O(1) in the sense of naive dimensional analysis [9]

## 3. Unitarization

Our unitarization scheme corresponds to a simple bubble-sum with amplitude given by

$$t_I(s)=V_I(s)\bigg/\left(1-\frac{1}{\pi}\int\frac{V_I(s')\,ds'}{s'-s-i\epsilon}\right). \tag{3.1}$$

In order to avoid double counting of graphs, the effective potential $V_I$ consists of all two-particle irreducible diagrams in the chiral expansion, with the exception of those contact graphs which are related to direct-channel resonance exchange [#3]. This amplitude can be thought of as arising from the solution of a relativistic Lipmann–Schwinger equation [11]. The leading order effective potential yields

$$t_I(s)=\frac{\alpha_I(s)}{1+(\alpha_I s/\pi)[\log(-s/\mu^2)+R_I(\mu^2)]}. \tag{3.2}$$

The $R_I$ are obtained by expanding eq. (3.2), and matching against the *direct-channel* piece of the chiral expansion. By inspection of eq. (2.2) we find $R_0=-6(2C_1+C_2)$ and $R_2=R_1=-12C_2$ (the "complementarity" between the $I=1$ and $I=2$ channels that follows from $R_2=R_1$ is investigated elsewhere in detail [10,12]).

The approximation of neglecting crossed-channel contributions clearly works best near an $s$-channel pole. Eq. (3.2) is therefore ideally suited to the task of investigating the existence of resonances for definite values of $C_1$ and $C_2$. More importantly, we find that if one wants to play the unitarization game and not be burdened with a large number of undetermined parameters, then the constraint that $S$-matrix elements should not be uniquely determined by elastic unitarity alone appears to require one to make this approximation. That axiomatic constraints like unitarity and causality do not uniquely determine $S$-matrix elements was an important lesson learned with the advent of QCD. A priori, there are an infinite number of $S$-matrices consistent with the most general physical principles [7]. (For example, in the context of a non-abelian gauge field theory, changing gauge group and fermion content certainly does not affect the unitarity of the theory.) In the present context, we translate this dictum into the constraint that predictions should always be accounted for by means independent of the unitarization scheme being used. For example, we see in eq. (3.2) that if $t_1$ is resonant, the width of the resonance is automatically fixed to the weak scale analogue of the KSRF relation [13]. However, we need not worry. This prediction is not a consequence of imposing elastic unitarity, but rather of neglecting the left-hand cut. This is easily seen by going to higher order in the effective potential. To order $s^2$ in the effective potential, we find

$$t_I(s)=\frac{\alpha_I s+\beta_I s^2[\log(s/\mu^2)+B_I(\mu^2)]}{1+(\alpha_I s/\pi)[\log(-s/\mu^2)+R_I(\mu^2)]+(\beta_I/2\pi)s^2\{[\log(-s/\mu^2)]^2+2B_I(\mu^2)\log(-s/\mu^2)+M_I(\mu^2)\}}, \tag{3.3}$$

where $\beta_0\equiv(-7/18\pi)(\alpha_0)^2$, $\beta_1\equiv(1/\pi)(\alpha_1)^2$, and $\beta_2\equiv(-11/9\pi)(\alpha_2)^2$ (see eq. (2.2).) The $B_I$ are the low-energy constants associated with heavy particle exchange in the crossed-channel. The $M_I$ are undetermined constants that appear at two-loop order in the chiral expansion. We see that it is by neglecting the contribution to the imaginary part of the inverse amplitude involving $B_1$ that we are able to predict the KSRF relation. Therefore, the predictive power of eq. (3.2) is not a result of imposing unitarity, but rather a result of neglecting a class of graphs associated with heavy particle exchanges in the crossed-channel, which are manifest at $O(s^2)$ in the chiral expansion. It is important to note that the above *does not* constitute a new derivation of the KSRF relation. In fact, all justifications of the KSRF relation, including the original current algebra derivation [8], *require* the tacit assumption that the left-hand cut of the $I=1$ scattering amplitude is effectively absent [14]. We find it powerful evidence in favor of our approach that, by ensuring that predictive power come from a source other than elastic unitarity, we arrive at a well-known derivation of the KSRF relation.

[#3] Eq. (3.1) is therefore of $N/D$ form in the sense that $V_I$ contains the left-hand cut. The right-hand cut is generated by the dispersion integral in the denominator function.

## 4. Case study. Intermediate states involving the standard model Higgs and heavy fermions of a hypothetical fourth generation doublet

The contributions to the low-energy constants $C_1$ and $C_2$ arising from intermediate states involving the Higgs boson and degenerate [#4] heavy fermions of a fourth generation doublet have been calculated perturbatively in ref. [15] using an on-shell subtraction scheme. They are given by [#5]

$$C_1^H(\mu)=\frac{1}{4}\left[\left(\frac{9\pi}{4\sqrt{3}}-\tfrac{37}{9}\right)-\tfrac{1}{6}\log\left(\frac{\mu^2}{M_H^2}\right)\right]+2\pi^2\left(\frac{v^2}{M_H^2}\right),\quad C_2^H(\mu)=\frac{1}{4}\left[-\left(\tfrac{2}{9}\right)-\tfrac{1}{3}\log\left(\frac{\mu^2}{M_H^2}\right)\right], \tag{4.1}$$

and

$$C_1^f=-\frac{N_c}{12}\left[\tfrac{1}{2}+6\left(\frac{2}{a}+\frac{4-a}{a^2}\int_0^1\log[1-ax(1-x)]\,dx\right)\right],\quad C_2^f=\frac{N_c}{12}, \tag{4.2}$$

where $a\equiv M_H^2/M_f^2$. Note that for definiteness we use values of the low-energy constants extracted from perturbation theory. However, we stress that we could equally well consider the most general couplings of fields with *any* quantum numbers to the Goldstone bosons, and estimate the values of these couplings using naive dimensional analysis. The uncertainty associated with a change of the $C_i$ of O(1) should certainly not exceed the inherent uncertainty that accompanies any unitarization scheme. In fact, we find that our basic conclusions are insensitive to natural changes in scale. For example, we can replace the $C_i^H$ by the values that obtain from coupling a scalar to the Goldstone bosons in the most general way [16]. In this case there is an undetermined parameter that can be related to the scalar width. If, instead of choosing the perturbative standard model value for the width, we choose one-half of that value, as is the case when the existence of a narrow p-wave resonance is assumed [13], our results are unaffected.

Inspection of eq. (4.2) reveals that the $I=1$ and $I=2$ amplitudes are independent of the heavy fermion mass. Only the $I=0$ amplitude has non-logarithmic contributions that depend on the Higgs and fermion masses. This is gratifying because it is these terms that are responsible for the "binding" which yields resonance behaviour. Nevertheless, this result is not surprising; the values of $C_1$ and $C_2$ given in eq. (4.2) are the low-energy manifestation of a scalar-dominated theory. Unitarization simply restores the basic properties of the assumed underlying theory.

In fig. 1 we schematically depict the complex $s$-plane. With a rather conservative choice of cutoff, given by $\Lambda=4\pi v\simeq$ (3 TeV), and with $M_H=M_f=1$ TeV, we see that the only pole in the theory is the "physical" Higgs boson. In fig. 2 we display the partial wave amplitudes of definite custodial isospin for values of the tree Higgs mass of 0.75 TeV and 1 TeV. For values of $M_f$ above 250 GeV, the fermionic contributions to the $I=0$ amplitude amount to a negligible renormalization of the physical Higgs mass, and so we neglect them in the graph. The complementary character of the non-resonant $I=1$ and $I=2$ amplitudes is clearly evident. We also display the Padé prediction for the $I=1$ amplitude, with $M_H=0.75$ TeV and $M_f=1$ TeV.

What is the source of discrepancy between our method and the Padé method? The method of Padé approximants, as applied in refs. [1–4], also predicts the KSRF relation in the $I=1$ channel, and yet the $O(s^2)$ crossed-channel contributions are *included*. Since the neglect of crossed-channel contributions can no longer serve as the source of predictive power, it follows that the crossed-channel contributions necessarily appear in the wrong place. These misplaced contributions appear in the real part of the inverse amplitude and therefore, in the current context, it is quite understandable that an unphysical pole is present. The moral of this story is that the Padé method, which is ideally suited to problems in potential theory, should probably not be applied to problems where crossing symmetry is important.

[#4] The heavy fermions are taken to be degenerate in order to avoid introducing isospin breaking terms.
[#5] We have corrected the error in $C_1^H(\mu)$ in ref. [15]

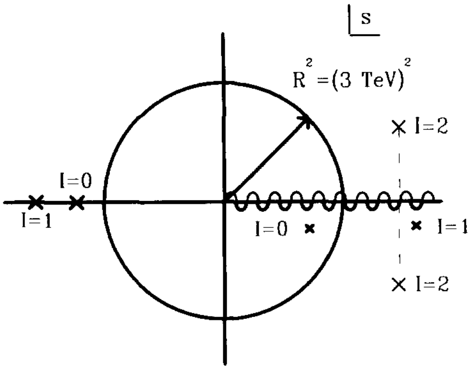
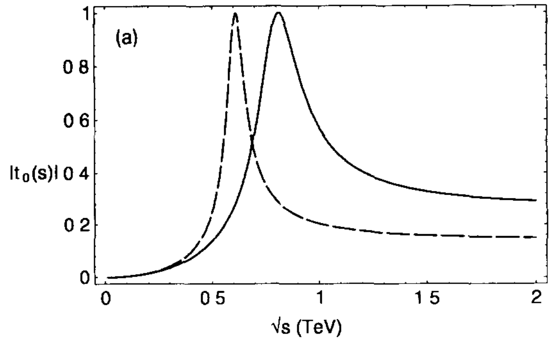
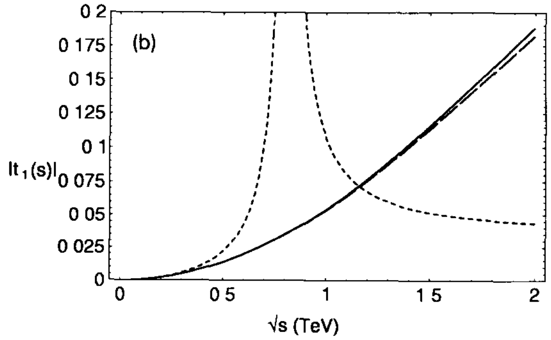



Fig. 1. Schematic depiction of the complex $s$-plane for characteristic values of the input parameters, $M_H=M_f=1$ TeV. The only pole below the cutoff is the physical Higgs pole.



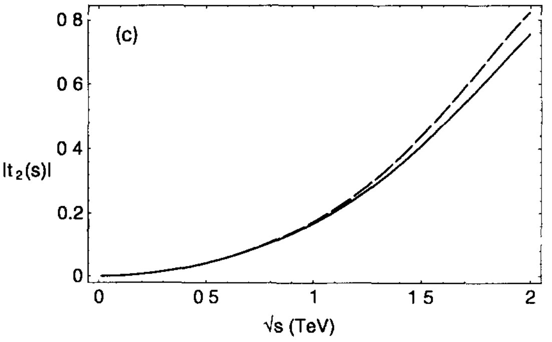




Fig. 2. (a) $I=0$ s-wave amplitude. The dashed line corresponds to $M_H=0.75$ TeV and the solid line to $M_H=1$ TeV. (b) $I=1$ p-wave amplitude. The dashed line corresponds to $M_H=0.75$ TeV and the solid line to $M_H=1$ TeV. The dotted line corresponds to the Padé method prediction for $M_H=0.75$ TeV and $M_f=1$ TeV. (c) $I=2$ s-wave amplitude. The dashed line corresponds to $M_H=0.75$ TeV and the solid line to $M_H=1$ TeV.

## 5. Conclusion

In this letter we argued that in a consistent unitary approximation, unique predictions of $S$-matrix elements should be accounted for independent of the specific unitarization scheme used. This constraint follows from the well-known fact that axiomatic principles like unitarity, while possessing constraining power, are devoid of predictive power. We constructed a unitary approximation for strong $W_L W_L$ scattering that satisfies this constraint, and yet possesses predictive power as a result of neglecting classes of graphs associated with crossed-channel physics, which in certain cases can be assumed small. In this sense our analysis is somewhat in the spirit of large $N$. For example, the $O(N)$ model is exactly solvable to leading order in $1/N$ precisely because left-hand cut contributions first appear at $O(1/N^2)$ [17]. In order to test the implications of our unitary parametrization, we considered the specific case of intermediate states involving a Higgs and heavy fermions of a hypothetical fourth generation doublet. We investigated the singularity structure of our unitary amplitudes, and found that the only nearby pole corresponds to the physical Higgs. We then showed that the Padé method prediction of a p-wave resonance can be understood based on the fact that the [1,1] Padé approximant does not satisfy the constraint outlined above. That is, certain predictions of the [1,1] approximant were shown to be artifacts of the unitarization scheme.

Our conclusions are not surprising. The effective field theory viewpoint implies that one gets out essentially what one puts in. Once we saturate the low-energy constants of chiral perturbation theory with contributions from a scalar-dominated underlying theory, information regarding the intermediate-energy spectrum is, in a sense, exhausted. The elastic unitarity constraint does not, and should not, change the character of the assumed underlying theory, albeit a strongly interacting one, e.g., by inducing a prominent vector contribution.

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