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Quantum scattering theory in the light of an exactly solvable model with rearrangement collisions

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QUANTUM SCATTERING THEORY IN THE LIGHT OF AN EXACTLY SOLVABLE MODEL WITH REARRANGEMENT COLLISIONS

by

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QUANTUM SCATTERING THEORY IN THE LIGHT OF AN EXACTLY SOLVABLE MODEL WITH REARRANGEMENT COLLISIONS

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We present an exactly solvable quantum field theory which allows rearrangement collisions. We solve the model and demonstrate the orthonormality and completeness of the solutions, and construct the S-matrix. In the light of the exact solutions constructed, we discuss various issues and assumptions in quantum scattering theory, including the isometry of the Möller wave matrix, the normalization and completeness of asymptotic states, and the non-orthogonality of basis states. We show that these common assertions do not obtain in this model. The model itself is sufficiently general that it could be easily used to study physical rearrangement processes.

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Chapter 1.
Introduction and Previous Work

Introduction

Quantum scattering has been an important subject of study since the early days of quantum physics. Unfortunately, while we have great understanding and intuition for simple scattering problems, such as single channel scattering, we cannot say the same for more general scattering problems such as multiple channel scattering, rearrangement collisions, field theoretic scattering, problems where bound states appear, and the like. There have been many attempts to generalize scattering theory to deal with more complicated cases. However, the literature in this field, though vast, is highly implicit. Most authors that have dealt with the problem have carried over intuition developed from the study of single channel potential scattering. This intuition, while quite adequate for simple problems, is ill-equipped to deal with more complicated scattering problems. Therefore, it is important to examine the common claim by some authors, for example Haag [1,2], that their formalism is general enough to encompass complicated scattering problems, as well as field theory. Unfortunately, most such formalisms are based largely on previous results from potential theory. Furthermore, even when these problems are addressed in quantum mechanical scattering, field theoretic scattering remains problematical. Many papers, such as the paper by Gell-Mann and Goldberger [3], treat field theoretic scattering as somewhat of an afterthought, without much development from first principles, or such as the papers by Van Hove [4], treat it as a case for discussion without any formalism. The first clear development of field theoretic scattering from first principles was the seminal paper by Lehmann, Symanzik, and Zimmerman [5]. However, contrary to usual belief, the LSZ formalism is not applicable in many cases, for example, collisions in which stable bound states appear. This is, in fact, pointed out by the authors themselves.

All this leads to the question: how many of our results and assumptions, and how much of our intuition can we carry over from simple single-channel potential scattering to more complicated scattering situations? To attempt to answer this question, we will construct and solve an exactly solvable quantum field theory. This model has a three particle sector, and allows rearrangement collisions. We will use the solutions of this model, along with previous results, to point out where the existing formalism has defects and shortcomings.

Our model, which we shall call the Rearrangement Model, is based on the Lee Model [6], but with extra particles and couplings chosen in such a way as to allow rearrangement collisions. The couplings of the model are $B \leftrightarrow C\phi$ and $D \leftrightarrow C\theta$. This model has a sector, which we shall call the Rearrangement Sector, $B\theta \leftrightarrow C\theta\phi \leftrightarrow D\phi$, in which rearrangement collisions can take place. The model is intrinsically useful, as noted in the abstract, because it is sufficiently general that it can be applied directly to physical problems involving rearrangement collisions. We shall, however, leave the applications to subsequent work.

We will construct the solutions of this model, and then, in the light of the solutions we have constructed, will examine various assumptions and assertions made in the literature about quantum scattering theory. In particular, we will focus on four key points, that of the isometry [1] of the

Möller wave matrix [7], the normalization [1] and completeness [8] of the asymptotic states, and the non-orthogonality of the physical $B\theta$, $C\theta\phi$, and $D\phi$ states [9,10]. We shall show that these assertions do not obtain in this model. We also comment upon the use of the (renormalized) free Hamiltonian in the literature [1,3,8,9,11], rather than the correct prescription, which is to use the comparison Hamiltonian (see Chapter 4). We shall also discuss the issues of redundant poles in the S-matrix and discrete states degenerate with the continuum.

The plan of this work is as follows. In the rest of this chapter we review some of the relevant earlier work. In Chapter 2 we introduce the Hamiltonian of the Rearrangement Model, show how explicit solutions can be found for this model in the Rearrangement Sector, and verify that the solutions obtained are, in fact, solutions to our model. In Chapter 3 we show that the solutions obtained are orthonormal and complete, write down the Möller Matrix and the comparison Hamiltonian, and show that the comparison Hamiltonian is isospectral with the full Hamiltonian, but not with the free Hamiltonian. Then we calculate the S-matrix of the system in this sector, demonstrate its unitarity, and calculate its eigenphases. In Chapter 4 we discuss scattering theory and its relation to the solutions that we constructed, and to previous work. Finally, in Chapter 5, we summarize our work and present our conclusions.

Redundant Poles and Phase Equivalent Potentials

We start by considering whether there exist non-relativistic systems for which the S-matrix has poles corresponding to redundant states, that is, states that are unnecessary for the completeness relation. We answer this question in the affirmative, and find that these states do not satisfy the Heisenberg relation. Furthermore, there exist phase equivalent systems in which these redundant poles of the S-matrix correspond to genuine bound states of the system, and must be included in the complete set of states. We will follow the development of Biswas, Pradhan, and Sudarshan [12].

The Heisenberg condition (which is essentially the statement that the poles of the S-matrix in the upper half plane of the complex momentum variable, *i.e.*the physical sheet of the complex energy variable, correspond to genuine bound states of the system; and a set of states must include these states before it is a complete set) is:

$$\int_{-\infty}^{\infty} dk\, S(k)\, e^{ikR} = \sum_n |c_n|^2 e^{-|k_n|R}. \tag{1.1}$$

If one considers a non-relativistic Schrödinger equation with an attractive potential $V(r) = -V_0 e^{-\frac{r}{a}}$ [13], one finds that this condition does not hold.

We can see this by constructing the S-matrix for this potential. The Schrödinger wave function vanishing at the origin is

$$u_k(r) = i\,(2\pi)^{-1/2} \left|\frac{\Gamma(1+\rho)}{J_{i\rho}(\alpha)}\right| \left[J_{-i\rho}\left(\alpha e^{-\frac{r}{2a}}\right) - J_{i\rho}(\alpha)\, J_{i\rho}\left(\alpha e^{-\frac{r}{2a}}\right)\right], \tag{1.2}$$

and the S-matrix is

$$S(k) = \frac{J_{i\rho}(\alpha)\,\Gamma(1+i\rho)}{J_{i\rho}(\alpha)\,\Gamma(1-i\rho)} \left(\frac{\alpha}{2}\right)^{-2i\rho}. \tag{1.3}$$

Here, the J's are Bessel functions, $\rho = 2ak$, and $\alpha = 2a\hbar^{-1}\sqrt{2mV_0}$.

The poles of the S-matrix come from two places, firstly, from $J_{|\rho|} = 0$, and secondly, from the poles of $\Gamma(1 + i\rho)$. The poles of Γ are the ones that do not correspond to genuine bound states. This is because even though the vanishing of $\Gamma(1+i\rho)$ allows non-trivial solutions of the Schrödinger equation, the wave function vanishes at these points. One can show that the completeness relation is satisfied without including these states. These states only manifest themselves through the Heisenberg condition which, for this potential, must be modified to [14]

$$\int_{-\infty}^{\infty} dk\, S(k)\, e^{ikR} = \sum_n |c_n|^2 e^{-|k_n|R} - 2\pi \sum_r \left(\frac{dS}{dk}\right)^{-1}\Bigg|_{k=-i|k_r|} e^{-|k_r|R}. \quad (1.4)$$

Furthermore, this is not all one can do. One can construct different potentials giving rise to the same S-matrix but with different completeness relations. For example, consider s-wave scattering by the following potentials:

$$a)\quad V_1(r) = -2\beta\lambda^2 \frac{e^{-\lambda r}}{\left(\beta e^{-\lambda r}+1\right)^2}, \quad 0>\beta>-1, \lambda>0, \qquad (1.5a)$$

$$b)\quad V_2(r) = \frac{6(r-\alpha)\left[(r-\alpha)^3 - 2\gamma^3\right]}{\left[(r-\alpha)^3+\gamma^3\right]^2}. \qquad (1.5b)$$

The normalized Schrödinger wave function vanishing at the origin is

$$\phi(k,r) = \frac{1}{2i|f(k)|}\left[f(k,0)\, f(-k,r) - f(-k,0)\, f(k,r)\right], \qquad (1.6)$$

where $f(k,r)$ is the Jost function with asymptotic behaviour

$$\lim_{r\to\infty} e^{ikr} f(k,r) = 1. \qquad (1.7)$$

The S-matrix then is

$$S(k) = \frac{f(k,0)}{f(-k,0)} \equiv \frac{f(k)}{f(-k)}. \qquad (1.8)$$

The Jost functions for these two potentials are found to be [13,15]:

$$f_1(k,r) = e^{-ikr}\frac{2k + i\lambda\left(\frac{\beta e^{-\lambda r}-1}{\beta e^{-\lambda r}+1}\right)}{2k - i\lambda}, \qquad (1.9a)$$

$$f_2(k,r) = e^{-ikr}\frac{4k^2 - \frac{12ik(r-\alpha)^2}{(r-\alpha)^3+\gamma^3} - \frac{12(r-\alpha)}{(r-\alpha)^3+\gamma^3}}{4k^2}. \qquad (1.9b)$$

With the proper choice of α and γ, both of these potentials lead to the same S-matrix,

$$S(k) = S_1(k) = S_2(k) = \frac{(2k + i\nu)(2k + i\lambda)}{(2k - i\lambda)(2k - i\nu)}, \tag{1.10}$$

where $\nu = \lambda\frac{\beta-1}{\beta+1} < 0$. This S-matrix has two poles on the imaginary axis for complex momenta: one in the lower half plane with $k = \frac{i}{2}\nu, \nu < 0$ and the other in the upper half plane with $k = \frac{i}{2}\lambda$. The latter pole should correspond to a bound state, but for the case of potential V_1, it does not because the wave function for V_1 vanishes at that point.

Furthermore, when we do the completeness integrals, we find that the residue of the integral for V_1 at $k = \frac{i}{2}\lambda$ vanishes, and thus the completeness condition for that case is just

$$\int_0^\infty \phi_1^*(kr)\,\phi_1(kr')\,dk = \delta(r - r'), \tag{1.11}$$

whereas for V_2 the completeness condition is

$$\int_0^\infty \phi_2^*(kr)\,\phi_2(kr')\,dk = \delta(r - r') - \phi_2^*\left(\frac{i}{2}\lambda, r\right)\phi_2\left(\frac{i}{2}\lambda, r'\right). \tag{1.12}$$

The Heisenberg condition for the first case, consistent with its completeness condition, should be

$$\int_{-\infty}^\infty S(k)\,e^{ikr}\,dk = 0, \tag{1.13}$$

whereas on actual computation one finds

$$\int_{-\infty}^\infty S(k)\,e^{ikr}\,dk = -4\pi\lambda\beta e^{\frac{-\lambda r}{2}}, \quad \beta < 0. \tag{1.14}$$

This violation is due to the presence of the redundant state. For the second potential, the Heisenberg condition is the same as the first potential, and is completely consistent with its completeness condition.

Another example is that of the model of [12]. This is a separable potential model, and the S-matrix of this model has two redundant poles. However, it is possible to construct a Lee Model that has the same S-matrix, but in which these poles correspond to genuine bound states, and are absolutely necessary for completeness. See [12] for details.

Discrete States Buried in the Continuum

We next consider whether it is possible to have a system which has one or more normalizable discrete states with energies that overlap the continuum. We again answer this question in the affirmative. We follow the development of Sudarshan [16].

A simple example of a system admitting one normalizable discrete state with an energy that overlaps the continuum is given by a Friedrichs-Lee type model. Its Hamiltonian is

$$H = {a_0}^\dagger a_0 m + \sum_j {a_j}^\dagger a_j w_j + \sum_j f_j \left({a_j}^\dagger a_0 + {a_0}^\dagger a_j \right). \tag{1.15}$$

The number operator

$$N = {a_0}^\dagger a_0 + \sum_j {a_j}^\dagger a_j \tag{1.16}$$

is a constant of the motion having non-negative integer eigenvalues. We restrict ourselves to the sector with $N = 1$, and specify the state with the amplitudes ψ_0 and ψ_j. We now let the index j take on continuous values with the corresponding natural frequencies taking the continuum of values $0 < W < \infty$. The states are then represented by a vector with a component ψ_0 and a function $\psi(W)$. The scalar product of two vectors ψ and ϕ is

$$(\psi, \phi) = \int_0^\infty dW \, \psi^* (W) \, \phi (W) + \psi_0^* \phi_0, \tag{1.17}$$

and the Hamiltonian acts on the vectors in the following manner:

$$(H\psi)_0 = m\psi_0 + \int_0^\infty dW\, f(W)\, \psi(W), \tag{1.18a}$$

$$(H\psi)(W) = f(W)\,\psi_0 + W\psi(W). \tag{1.18b}$$

We can now solve the model in the usual manner to get the continuum eigenvectors

$${}_\lambda\psi_0 = \frac{f(\lambda)}{\alpha(\lambda + i\epsilon)}, \tag{1.19a}$$

$${}_\lambda\psi(W) = \delta(\lambda - W) + \frac{f(W)f(\lambda)}{(\lambda - W + i\epsilon)\,\alpha(\lambda + i\epsilon)}, \tag{1.19b}$$

where

$$\alpha(Z) = Z - m - \int_0^\infty dW\, (Z - W)^{-1} f^2(W). \tag{1.20}$$

The model has a discrete state, as usual, if m is such that

$$\alpha(0) = -m + \int_0^\infty dW\, W^{-1} f^2(W) > 0. \tag{1.21}$$

Then there exists a real negative number, M, such that

$$\alpha(M) = 0, \tag{1.22}$$

and so we have a discrete state with the eigenvector

$${}_M\psi_0 = \frac{1}{\sqrt{\alpha'(M)}}, \tag{1.23a}$$

$${}_M\psi(W) = f(W)\frac{1}{(M - W)\sqrt{\alpha'(M)}}. \tag{1.23b}$$

The continuum solutions, Eqs. (1.19), are complete unless $\alpha(M) = 0$ provided that $f(W)$ is non-vanishing in $0 < W < \infty$. If such a zero exists, then we must include Eqs. (1.23) for completeness.

If $f(W)$ is non-vanishing for all W in the range $0 < W < \infty$ then we cannot have a discrete eigenvalue for $M > 0$. Since we want a discrete eigenvalue, we allow $f(W)$ to have one zero in this domain; consequently, $\alpha(Z)$ can have a zero in this range. Therefore, we take $f(W)$ to be such that

$$f(M) = 0, \quad \alpha(M) = 0, \tag{1.24}$$

and then there is a discrete solution of the form of Eqs. (1.23) with $M > 0$. This eigenvalue is degenerate; in addition to Eqs. (1.23) there is a non-normalizable solution belonging to the continuum with

$$\psi(W) = \delta(M - W), \quad \psi_0 = 0, \tag{1.25}$$

which corresponds to a plane wave solution.

This model admits only one such "buried" discrete solution because the real part of $\alpha'(Z)$ is non-negative along the real axis and so $\alpha(Z)$ can have at most one zero. If one is interested in more than one such state, one must generalize this model. Such a model is a straightforward generalization of Eq. (1.15) constructed by taking

$$H = \sum_{\alpha} a_\alpha{}^\dagger a_\alpha m_\alpha + \sum_j a_j{}^\dagger a_j W_j + \sum_{\alpha j} f_{\alpha j}\left(a_j{}^\dagger a_\alpha + a_\alpha{}^\dagger a_j\right), \tag{1.26}$$

and proceeding to the continuum limit, $0 < W < \infty$, for W_j. The states are now represented by vectors with discrete components ψ_α and a complex valued function $\psi(W)$. For this case, one finds simultaneous linear inhomogeneous equations for the eigenvectors

$$(\lambda - m_\alpha)\psi_\alpha = \sum_\beta F_{\alpha\beta}(\lambda + i\epsilon)\psi_\beta + f_\alpha(\lambda)\theta(\lambda), \tag{1.27a}$$

$$(\lambda - W)\psi(W) = \sum_\beta f_\beta(W)\psi_\beta, \tag{1.27b}$$

and the solutions can be found in terms of the determinants of the coefficients. The analogue of the α function defined in the first example is then a determinant,

$$\Delta(Z) = \det\left[Z\delta_{\alpha\beta} - m_\alpha\delta_{\alpha\beta} - F_{\alpha\beta}(Z)\right] = \det D_{\alpha\beta}(z). \tag{1.28}$$

The discrete solutions only arise when $\Delta(Z) = 0$ for $Z < 0$, and we can have up to the range of the index α of these. Now, to get the type of solution we want, we require that at any such solution, $\lambda = M > 0$, the coupling constants $f_\alpha(M)$ all vanish. Then, exactly as in the first example, we have a discrete solution whose eigenvector equations are

$$\sum_\alpha {}_M\psi_\alpha \frac{f_\alpha(W)}{(M - W)} = {}_M\psi(W), \tag{1.29}$$

$$\sum_\beta F_{\alpha\beta}(M)\, {}_M\psi_\beta = (\lambda - m_\alpha)\, {}_M\psi_\beta. \tag{1.30}$$

By suitable choices of parameters, we can have any number of discrete states buried in the continuum.

Still another generalization is obtained by choosing

$$H = \sum_j a_j{}^\dagger a_j W_j \pm \sum_{jk} f_j f_k \left(a_j{}^\dagger a_k + a_k^\dagger a_j\right), \tag{1.31}$$

with the $+$ sign corresponding to a repulsive potential and the $-$ sign to an attractive potential.

We can follow precisely the same analysis as before to see that there is a discrete solution buried in the continuum for the repulsive case. See [16] for details.

Chapter 2.
The Model

The Rearrangement Model

To keep contact with earlier work, we shall use a combination of the notations of [17] and [18], as far as possible. We consider a quantum field theory with five distinct fields, B, C, D, θ, and ϕ, and the corresponding particles (no anti-particles).

The non-zero commutators are:

$$\begin{aligned}&\left[B,B^\dagger\right]=\left[D,D^\dagger\right]=\left[C,C^\dagger\right]=1,\\&\left[\theta(\omega),\theta^\dagger(\omega')\right]=\delta\left(\omega-\omega'\right),\quad\left[\phi(\nu),\phi^\dagger(\nu')\right]=\delta\left(\nu-\nu'\right).\end{aligned}\tag{2.1}$$

Note that θ and ϕ are labelled by continuum parameters, $0<\omega,\nu<\infty$, while A, B, and C, are treated as single modes ("infinitely heavy") [19]. This assumption is made purely to simplify the kinematics of the problem; no essentially new results would obtain if we were to allow these modes to propagate. We choose to use the energy as our variable, rather than momentum, because this makes the model much simpler, and more physically transparent. In Appendix A we present a short review of the Lee Model in the usual momentum basis to make this point clear. We want a total Hamiltonian for the system which allows the transitions

$$B\leftrightarrow C\phi,$$

and

$$D \leftrightarrow C\theta.$$

Therefore, we choose our Hamiltonian to be:

$$H = H_0 + V, \tag{2.2}$$

where

$$H_0 = m_B B^\dagger B + m_D D^\dagger D + \int d\omega\, \omega \theta^\dagger(\omega)\theta(\omega) + \int d\nu\, \nu \phi^\dagger(\nu)\phi(\nu), \tag{2.3}$$

and

$$\begin{aligned} V = &\int d\omega\, f(\omega)\, \theta(\omega) C D^\dagger + \int d\omega\, f^*(\omega)\, \theta^\dagger(\omega) C^\dagger D \\ &+ \int d\nu\, g(\nu)\, \phi(\nu) C B^\dagger + \int d\nu\, g^*(\nu)\, \phi^\dagger(\nu) C^\dagger B. \end{aligned} \tag{2.4}$$

This Hamiltonian has three constants of motion apart from itself. They are:

$$C_1 = B^\dagger B + C^\dagger C + D^\dagger D, \tag{2.5a}$$

$$C_2 = B^\dagger B + \int d\nu\, \phi^\dagger(\nu)\phi(\nu), \tag{2.5b}$$

$$C_3 = D^\dagger D + \int d\omega\, \theta^\dagger(\omega)\theta(\omega). \tag{2.5c}$$

Therefore, no transitions can occur between sectors labelled by these quantum numbers. Let us start by enumerating the stable sectors. The first such sector is the vacuum and has $C_1 = C_2 = C_3 = 0$. The next three are: $C_1 = 1, C_2 = 0, C_3 = 0$; $C_1 = 0, C_2 = 1, C_3 = 0$; and $C_1 = 0, C_2 = 0, C_3 = 1$. These correspond to the states C, ϕ, and θ, respectively. Finally, there is the sector with $C_1 = 0, C_2 = 1, C_3 = 1$; it corresponds to a state $\theta\phi$.

The three lowest non-trivial sectors are:

$$C_1 = 1, C_2 = 1, C_3 = 0, \tag{2.6a}$$

$$C_1 = 1, C_2 = 0, C_3 = 1, \tag{2.6b}$$

$$C_1 = 1, C_2 = 1, C_3 = 1. \tag{2.6c}$$

These correspond to $B \leftrightarrow C\phi$, $D \leftrightarrow C\theta$, and $B\theta \leftrightarrow C\theta\phi \leftrightarrow D\phi$, respectively. The last of these is the sector in which rearrangement collisions can take place, and as mentioned before, we shall call this the Rearrangement Sector.

Our strategy for solving the model in the Rearrangement Sector will be to first construct the solutions of the two lowest non-trivial sectors, (2.6a) and (2.6b) (which are exactly analogous to the Lee Model), and then use these solutions to expand the Rearrangement Sector equations.

Solving the Model

We start by constructing the solutions for the $B \leftrightarrow C\phi$ and $D \leftrightarrow C\theta$ sectors. These are exactly the same as the Lee model, so the solutions are simple. We shall denote non-interacting ("bare") states by single bras and kets ($\langle\ ,\ \rangle$), and interacting states ("dressed" or "physical") by double bras and kets ($\langle\langle\ ,\ \rangle\rangle$).

The equations we need to solve for the continuum solutions are

$$H|\lambda\rangle\rangle = \lambda|\lambda\rangle\rangle \tag{2.7a}$$

in the $B \leftrightarrow C\phi$ sector, and

$$H|\mu\rangle\rangle = \mu|\mu\rangle\rangle \tag{2.7b}$$

in the $D \leftrightarrow C\theta$ sector.

We define

$$\alpha(z) \equiv z - m_D - \int_0^\infty d\omega \, \frac{|f(\omega)|^2}{z-\omega}, \tag{2.8}$$

$$\beta(z) \equiv z - m_B - \int_0^\infty d\nu \, \frac{|g(\nu)|^2}{z-\nu}, \tag{2.9}$$

$$\rho_{\lambda,B}(\nu) \equiv \langle C\phi(\nu)|\lambda\rangle\rangle, \tag{2.10}$$

$$\rho_{\mu,D}(\omega) \equiv \langle C\theta(\omega)|\mu\rangle\rangle, \tag{2.11}$$

$$\sigma_{\lambda,B} \equiv \langle B|\lambda\rangle\rangle, \tag{2.12}$$

$$\sigma_{\mu,D} \equiv \langle D|\mu\rangle\rangle. \tag{2.13}$$

For shorthand, we will write $\alpha(\lambda)$ for $\alpha(\lambda + i\epsilon)$ and $\alpha^*(\lambda)$ for $\alpha(\lambda - i\epsilon)$, and similarly for $\beta(\lambda)$. In terms of these, the solutions are:

$$\rho_{\lambda,B}(\nu) = \delta(\lambda - \nu) + \frac{g^*(\nu)\,\sigma_{\lambda,B}}{\lambda - \nu + i\epsilon}, \tag{2.14a}$$

$$\rho_{\mu,D}(\omega) = \delta(\mu - \omega) + \frac{f^*(\omega)\,\sigma_{\mu,D}}{\mu - \omega + i\epsilon}, \tag{2.14b}$$

$$\sigma_{\lambda,B} = \frac{g(\lambda)}{\beta(\lambda)}, \tag{2.14c}$$

$$\sigma_{\mu,D} = \frac{f(\mu)}{\alpha(\mu)}. \tag{2.14d}$$

If $\alpha(z)$ develops zeros then we have additional discrete states. Similarly, if $\beta(z)$ develops zeros then again we have additional discrete states. For our purposes, we shall always assume that both $\alpha(z)$ and $\beta(z)$ have exactly one zero each, which are denoted by M_D and M_B, respectively. There is no loss of generality if we use this assumption because the extension to more than one zero is trivial. The equations for the discrete states are:

$$H|M_B\rangle\rangle = M_B|M_B\rangle\rangle \tag{2.15a}$$

in the $B \leftrightarrow C\phi$ sector, and

$$H|M_D\rangle\rangle = M_D|M_D\rangle\rangle \tag{2.15b}$$

in the $D \leftrightarrow C\theta$ sector. We define

$$\rho_B(\nu) \equiv \langle C\phi(\nu)|M_B\rangle\rangle, \tag{2.16}$$

$$\rho_D(\omega) \equiv \langle C\theta(\omega)|M_D\rangle\rangle, \tag{2.17}$$

$$Z_B \equiv \langle B|M_B\rangle\rangle, \tag{2.18}$$

$$Z_D \equiv \langle D|M_D\rangle\rangle. \tag{2.19}$$

In terms of these, the normalized solutions are:

$$\rho_B(\nu) = Z_B \frac{g^*(\nu)}{M_B - \nu}, \tag{2.20a}$$

$$\rho_D(\omega) = Z_D \frac{f^*(\omega)}{M_D - \omega}, \tag{2.20b}$$

$$|Z_B|^2 = \left[1 + \int d\nu \, \frac{|g(\nu)|^2}{(M_B - \nu)^2}\right]^{-1}, \tag{2.20c}$$

$$|Z_D|^2 = \left[1 + \int d\omega \, \frac{|f(\omega)|^2}{(M_D - \omega)^2}\right]^{-1}, \tag{2.20d}$$

where the last two are obtained by imposition of the orthonormality condition. Note that these solutions, Eqs. (2.14) and (2.20), form a complete orthonormal set.

Now, we use these solutions to construct the solutions in the Rearrangement Sector. In this sector we will have four sorts of solutions. The first will correspond to the "physical" $|C\theta(\omega)\phi(\nu)\rangle\rangle$ sector, the second to the "physical" $|D\phi(\nu)\rangle\rangle$, the third to the "physical" $|B\theta(\omega)\rangle\rangle$, and the last to one or more dynamically generated bound states, which we shall denote by $|M_A\rangle\rangle$. Which solution is obtained will depend on where we put the delta functions at infinity (which represent the plane wave parts of our solutions) in our solutions. If we put none, we get the discrete states.

We wish to solve the eigenvalue equation

$$H|E\rangle\rangle = E|E\rangle\rangle. \tag{2.21}$$

We expand Eq. (2.21) in terms of $\theta^\dagger(\omega)|\lambda\rangle\rangle$, $\theta^\dagger(\omega)|M_B\rangle\rangle$, $\phi^\dagger(\nu)|\mu\rangle\rangle$, and $\phi^\dagger(\nu)|M_D\rangle\rangle$. By acting on these states with the Hamiltonian, and using Eqs. (2.7) and (2.15) we get

$$(E - \lambda - \omega)\langle\langle\lambda|\theta(\omega)|E\rangle\rangle = f^*(\omega)\,\langle\langle\lambda|C^\dagger D|E\rangle\rangle, \tag{2.22a}$$

$$(E - M_B - \omega)\langle\langle M_B|\theta(\omega)|E\rangle\rangle = f^*(\omega)\,\langle\langle M_B|C^\dagger D|E\rangle\rangle, \tag{2.22b}$$

$$(E-\mu-\nu)\langle\langle\mu|\phi(\nu)|E\rangle\rangle = g^*(\nu)\,\langle\langle\mu|C^\dagger B|E\rangle\rangle, \tag{2.22c}$$

$$(E-M_D-\nu)\langle\langle M_D|\phi(\nu)|E\rangle\rangle = g^*(\nu)\,\langle\langle M_D|C^\dagger B|E\rangle\rangle. \tag{2.22d}$$

We need to evaluate the unknown matrix elements on the right hand side of Eqs. (2.22). We solve for these elements by inserting H in them, commuting it on one side, and letting it act on $|E\rangle\rangle$ on the other. For example, we can solve for $\langle\langle\lambda|C^\dagger D|E\rangle\rangle$ in the following manner:

$$\begin{aligned}&\langle\langle\lambda|C^\dagger DH|E\rangle\rangle = E\langle\langle\lambda|C^\dagger D|E\rangle\rangle\\ &\quad\Rightarrow \langle\langle\lambda|\Big\{HC^\dagger D+\Big[C^\dagger D,H\Big]\Big\}|E\rangle\rangle = E\langle\langle\lambda|C^\dagger D|E\rangle\rangle.\end{aligned} \tag{2.23}$$

We now let H in the first term of Eq. (2.23) act on $\langle\langle\lambda|$, and evaluate the commutator in the second term. We proceed similarly for the other three equations and, when all the dust settles, get

$$\begin{aligned}&(E-\lambda-m_D)\,\langle\langle\lambda|CD^\dagger|E\rangle\rangle\\ &\quad=\int d\omega\, f(\omega)\,\langle\langle\lambda|\theta(\omega)|E\rangle\rangle\\ &\quad-\sigma_{\lambda,B}{}^*\Big[\int d\omega\, f(\omega)\Big\{\int d\lambda'\sigma_{\lambda',B}\langle\langle\lambda'|\theta(\omega)|E\rangle\rangle + Z_B\langle\langle M_B|\theta(\omega)|E\rangle\rangle\Big\}\\ &\qquad\quad+\int d\nu\, g(\nu)\Big\{\int d\mu'\sigma_{\mu',D}\langle\langle\mu'|\phi(\nu)|E\rangle\rangle + Z_D\langle\langle M_D|\phi(\nu)|E\rangle\rangle\Big\}\Big],\end{aligned} \tag{2.24a}$$

$$\begin{aligned}&(E-M_B-m_D)\,\langle\langle M_B|CD^\dagger|E\rangle\rangle\\ &\quad=\int d\omega\, f(\omega)\,\langle\langle M_B|\theta(\omega)|E\rangle\rangle\\ &\quad-Z_B^*\Big[\int d\omega\, f(\omega)\Big\{\int d\lambda'\sigma_{\lambda',B}\langle\langle\lambda'|\theta(\omega)|E\rangle\rangle + Z_B\langle\langle M_B|\theta(\omega)|E\rangle\rangle\Big\}\\ &\qquad\quad+\int d\nu\, g(\nu)\Big\{\int d\mu'\sigma_{\mu',D}\langle\langle\mu'|\phi(\nu)|E\rangle\rangle + Z_D\langle\langle M_D|\phi(\nu)|E\rangle\rangle\Big\}\Big],\end{aligned} \tag{2.24b}$$

$$
\begin{aligned}
&(E-\mu-m_B)\,\langle\langle\mu|CB^\dagger|E\rangle\rangle \\
&\quad=\int d\nu\, g(\nu)\,\langle\langle\mu|\phi(\nu)|E\rangle\rangle \\
&\quad-\sigma_{\mu,D^*}\Big[\int d\omega\, f(\omega)\Big\{\int d\lambda'\sigma_{\lambda',B}\langle\langle\lambda'|\theta(\omega)|E\rangle\rangle+Z_B\langle\langle M_B|\theta(\omega)|E\rangle\rangle\Big\} \\
&\qquad+\int d\nu\, g(\nu)\Big\{\int d\mu'\sigma_{\mu',D}\langle\langle\mu'|\phi(\nu)|E\rangle\rangle+Z_D\langle\langle M_D|\phi(\nu)|E\rangle\rangle\Big\}\Big],
\end{aligned}
\tag{2.24c}
$$

$$
\begin{aligned}
&(E-M_D-m_B)\,\langle\langle M_D|CB^\dagger|E\rangle\rangle \\
&\quad=\int d\nu\, g(\nu)\,\langle\langle M_D|\phi(\nu)|E\rangle\rangle \\
&\quad-Z_D^*\Big[\int d\omega\, f(\omega)\Big\{\int d\lambda'\sigma_{\lambda',B}\langle\langle\lambda'|\theta(\omega)|E\rangle\rangle+Z_B\langle\langle M_B|\theta(\omega)|E\rangle\rangle\Big\} \\
&\qquad+\int d\nu\, g(\nu)\Big\{\int d\mu'\sigma_{\mu',D}\langle\langle\mu'|\phi(\nu)|E\rangle\rangle+Z_D\langle\langle M_D|\phi(\nu)|E\rangle\rangle\Big\}\Big].
\end{aligned}
\tag{2.24d}
$$

We first solve for the "physical" $|C\theta(\omega)\phi(\nu)\rangle\rangle$ states. We start by inverting Eqs. (2.22) and putting in the requisite delta functions at infinity. Note that we have to put in two delta functions because this is an infinitely degenerate (double) continuum, and cannot be specified by just E; rather, we have to label the state with the variables E and n, with the n variable representing the division of energy between the θ and ϕ particles. We then substitute these into the first term of each of Eqs. (2.24), and solve for the unknown matrix elements. Having found them, we put them into Eqs. (2.22) to find our solutions. Defining

$$b^C(E,n,\lambda,\omega)\equiv\langle\langle\lambda|\theta(\omega)|E,n\rangle\rangle, \tag{2.25a}$$

$$b_F^C(E,n,M_B,\omega)\equiv\langle\langle M_B|\theta(\omega)|E,n\rangle\rangle, \tag{2.25b}$$

$$d^C(E,n,\mu,\nu)\equiv\langle\langle\mu|\phi(\nu)|E,n\rangle\rangle, \tag{2.25c}$$

$$d_F^C(E,n,M_D,\nu)\equiv\langle\langle M_D|\phi(\nu)|E,n\rangle\rangle, \tag{2.25d}$$

we get

$$
\begin{aligned}
b^{C}(E,n,\lambda,\omega) &= \delta(E-\lambda-n)\rho_{n,D}(\omega) \\
&\qquad - \frac{f^{*}(\omega)}{(E-\lambda-\omega+i\epsilon)}\frac{\sigma_{\lambda,B}{}^{*}}{\alpha(E-\lambda)}K_C(E,n), \qquad (2.26a) \\
b^{C}_{F}(E,n,M_B,\omega) &= -\frac{f^{*}(\omega)}{(E-M_B-\omega+i\epsilon)}\frac{Z_B^{*}}{\alpha(E-M_B)}K_C(E,n), \qquad (2.26b) \\
d^{C}(E,n,\mu,\nu) &= \delta(\mu-n)\,\rho_{E-n,B}(\nu) \\
&\qquad - \frac{g^{*}(\nu)}{(E-\mu-\nu+i\epsilon)}\frac{\sigma_{\mu,D}{}^{*}}{\beta(E-\mu)}K_C(E,n), \qquad (2.26c) \\
d^{C}_{F}(E,n,M_D,\nu) &= -\frac{g^{*}(\nu)}{(E-M_D-\nu+i\epsilon)}\frac{Z_D^{*}}{\beta(E-M_D)}K_C(E,n), \qquad (2.26d)
\end{aligned}
$$

where

$$
K_C(E,n) = \frac{g(E-n)\,f(n)}{\beta(E-n)\,\alpha(n)}\frac{1}{\gamma(E)}, \qquad (2.27)
$$

$$
\begin{aligned}
\gamma(E) &= \frac{|Z_D|^2}{\beta(E-M_D)} + \int d\mu\,\frac{|f(\mu)|^2}{|\alpha(\mu)|^2}\frac{1}{\beta(E-\mu)} \qquad (2.28a) \\
&= \frac{|Z_B|^2}{\alpha(E-M_B)} + \int d\lambda\,\frac{|g(\lambda)|^2}{|\beta(\lambda)|^2}\frac{1}{\alpha(E-\lambda)}. \qquad (2.28b)
\end{aligned}
$$

The last equality follows from Eqs. (B.10).

We now solve for the "physical" $|D\phi(\nu)\rangle\rangle$ sector. We again start by inverting Eqs. (2.22), but this time substituting in just the one requisite delta function at infinity in Eq. (2.22d). We put these equations in Eqs. (2.24), and solve for the unknown matrix elements putting in another delta function at infinity in Eq. (2.24a) when inverting because now we must account for the zero of $\alpha(E-\lambda)$ at M_D. We then put these results in Eqs. (2.22). Defining

$$
\begin{aligned}
b^{D}(E,\lambda,\omega) &\equiv \langle\langle\lambda|\theta(\omega)|E\rangle\rangle, \qquad (2.29a) \\
b^{D}_{F}(E,M_B,\omega) &\equiv \langle\langle M_B|\theta(\omega)|E\rangle\rangle, \qquad (2.29b) \\
d^{D}(E,\mu,\nu) &\equiv \langle\langle\mu|\phi(\nu)|E\rangle\rangle, \qquad (2.29c) \\
d^{D}_{F}(E,M_D,\nu) &\equiv \langle\langle M_D|\phi(\nu)|E\rangle\rangle, \qquad (2.29d)
\end{aligned}
$$

we get

$$
\begin{aligned}
b^D(E,\lambda,\omega) &= \rho_D(\omega)\,\delta(E-\lambda-M_D) \\
&\quad - \frac{f^*(\omega)}{(E-\lambda-\omega+i\epsilon)}\frac{\sigma_{\lambda,B}{}^*}{\alpha(E-\lambda)}K_D(E), && (2.30a) \\
b_F^D(E,M_B,\omega) &= -\frac{f^*(\omega)}{(E-M_B-\omega+i\epsilon)}\frac{Z_B^*}{\alpha(E-M_B)}K_D(E), && (2.30b) \\
d^D(E,\mu,\nu) &= -\frac{g^*(\nu)}{(E-\mu-\nu+i\epsilon)}\frac{\sigma_{\mu,D}{}^*}{\beta(E-\mu)}K_D(E), && (2.30c) \\
d_F^D(E,M_D,\nu) &= \rho_{E-M_D,B}(\nu) \\
&\quad - \frac{g^*(\nu)}{(E-M_D-\nu+i\epsilon)}\frac{Z_D^*}{\beta(E-M_D)}K_D(E), && (2.30d)
\end{aligned}
$$

where

$$
K_D(E) = \frac{Z_D}{\gamma(E)}\frac{g(E-M_D)}{\beta(E-M_D)}, \tag{2.31}
$$

and $\gamma(E)$ is the same as that defined in Eqs. (2.28).

In exactly the same way, we can find the solutions for the "physical" $|B\theta(\omega)\rangle\rangle$ sector. They are:

$$
\begin{aligned}
b^B(E,\lambda,\omega) &= -\frac{f^*(\omega)}{(E-\lambda-\omega+i\epsilon)}\frac{\sigma_{\lambda,B}{}^*}{\alpha(E-\lambda)}K_B(E), && (2.32a) \\
b_F^B(E,M_B,\omega) &= \rho_{E-M_B,D}(\omega) \\
&\quad - \frac{f^*(\omega)}{(E-M_B-\omega+i\epsilon)}\frac{Z_B^*}{\alpha(E-M_B)}K_B(E), && (2.32b) \\
d^B(E,\mu,\nu) &= \rho_B(\nu)\,\delta(E-\mu-M_B) \\
&\quad - \frac{g^*(\nu)}{(E-\mu-\nu+i\epsilon)}\frac{\sigma_{\mu,D}{}^*}{\beta(E-\mu)}K_B(E), && (2.32c) \\
d_F^B(E,M_D,\nu) &= -\frac{g^*(\nu)}{(E-M_D-\nu+i\epsilon)}\frac{Z_D^*}{\beta(E-M_D)}K_B(E), && (2.32d)
\end{aligned}
$$

where

$$
K_B(E) = \frac{Z_B}{\gamma(E)}\frac{f(E-M_B)}{\alpha(E-M_B)}, \tag{2.33}
$$

and $\gamma(E)$ is the same as that defined in Eq. (2.28).

Finally, we wish to solve for any dynamically generated discrete states. In this case, we put no delta functions anywhere. When we follow the procedure of putting Eqs. (2.22) in Eqs. (2.24) and solving for the unknown matrix elements, we find that the only way to satisfy all the equations is if $\gamma(E)$ has zeros. Denoting these zeros by M_A, we find the discrete state solutions:

$$b^A(M_A,\lambda,\omega) = -\frac{f^*(\omega)}{(M_A-\lambda-\omega)}\frac{\sigma_{\lambda,B}{}^*}{\alpha(M_A-\lambda)}K_A(M_A), \qquad (2.34a)$$

$$b_F^A(M_A,M_B,\omega) = -\frac{f^*(\omega)}{(M_A-M_B-\omega+i\epsilon)}\frac{Z_B^*}{\alpha(M_A-M_B)}K_A(M_A), \qquad (2.34b)$$

$$d^A(M_A,\mu,\nu) = -\frac{g^*(\nu)}{(M_A-\mu-\nu)}\frac{\sigma_{\mu,D}{}^*}{\beta(M_A-\mu)}K_A(M_A), \qquad (2.34c)$$

$$d_F^A(M_A,M_D,\nu) = -\frac{g^*(\nu)}{(M_A-M_D-\nu+i\epsilon)}\frac{Z_D^*}{\beta(M_A-M_D)}K_A(M_A), \qquad (2.34d)$$

where $K_A(M_A)$ is now an arbitrary normalization factor which is fixed, when demonstrating completeness, to be $\sqrt{\frac{d\gamma(E)}{dE}|_{E=M_A}}$ (see the discussion after Eq. (3.18)). For our purposes, without loss of generality, we assume that there is only one zero of $\gamma(E)$, denoted by M_A, and thus only one dynamically generated discrete state. The extension to more than one discrete state is trivial.

In each of Eqs. (2.26), (2.30), (2.32), and (2.34) the superscript refers to the sector that the solution is in, and the subscript F refers to solutions expanded in the the discrete state part of the Lee Model sectors. Furthermore, we have anticipated future developments by fixing the arbitrary constants accompanying the delta functions in Eqs. (2.26), (2.30), and (2.32). We do this by demanding that Eq. (2.37a) and Eq. (2.37b), or their equivalents for the other two sectors, give the same result, and that the solutions be orthonormal.

Verification of the Solutions

We now proceed to verify that Eqs. (2.26), (2.30), (2.32), and (2.34) are each solutions to our problem. To do this, we first transform our solutions into the bare state basis; *i.e.* in terms of the non-interacting states $|C\theta(\omega)\phi(\nu)\rangle$, $|B\theta(\omega)\rangle$, and $|D\phi(\nu)\rangle$, using the completeness of the lower sector solutions. With the expansion coefficients in the "physical" $|C\theta(\omega)\phi(\nu)\rangle\rangle$ sector defined in the following manner (with the coefficients for the other sectors defined similarly)

$$
\begin{aligned}
|E,n\rangle\rangle \equiv\ & C^C(E,n,\omega,\nu)|C\theta(\omega)\phi(\nu)\rangle \\
& + B^C(E,n,\omega)|B\theta(\omega)\rangle + D^C(E,n,\nu)|D\phi(\nu)\rangle,
\end{aligned} \tag{2.35}
$$

where

$$
\Psi^C(E,n,\omega,\nu) \equiv \begin{pmatrix} C^C(E,n,\omega,\nu) \\ D^C(E,n,\nu) \\ B^C(E,n,\omega) \end{pmatrix}, \tag{2.36}
$$

and

$$
\begin{aligned}
C^C(E,n,\omega,\nu) &\equiv \langle C\theta(\omega)\phi(\nu)|E,n\rangle\rangle \\
B^C(E,n,\omega) &\equiv \langle B\theta(\omega)|E,n\rangle\rangle \\
D^C(E,n,\nu) &\equiv \langle D\phi(\nu)|E,n\rangle\rangle.
\end{aligned}
$$

we have

$$
C^C(E,n,\omega,\nu) = \int d\lambda\, \rho_{\lambda,B}(\nu) b^C(E,n,\lambda,\omega) + \rho_B(\nu) b_F^C(E,n,M_B,\omega), \tag{2.37a}
$$

$$
= \int d\mu\, \rho_{\mu,D}(\omega) d^C(E,n,\mu,\nu) + \rho_D(\omega) d_F^C(E,n,M_D,\nu), \tag{2.37b}
$$

$$
D^C(E,n,\nu) = \int d\mu\, \sigma_{\mu,D} d^C(E,n,\mu,\nu) + Z_D d_F^C(E,n,M_D,\nu), \tag{2.37c}
$$

$$
B^C(E,n,\omega) = \int d\lambda\, \sigma_{\lambda,B} b^C(E,n,\lambda,\omega) + Z_B b_F^C(E,n,M_B,\omega), \tag{2.37d}
$$

with similar equations for the other three sectors (for example, for the "physical" $|D\phi(\nu)\rangle\rangle$ sector, we would replace $C^C(E,n,\omega,\nu)$ by $C^D(E,\omega,\nu)$, $b^C(E,n,\lambda,\omega)$ by $b^D(E,\lambda,\omega)$, etc.). A good check that we have solved our equations correctly is to verify that Eqs. (2.37a) and (2.37b) give the same result. This is indeed completely trivial if we use Eq. (B.14).

For the "physical" $|C\theta(\omega)\phi(\nu)\rangle\rangle$ sector in the bare basis, we get:

$$C^C(E,n,\omega,\nu) = \rho_{n,D}(\omega)\rho_{E-n,B}(\nu) - K_C(E,n)\frac{f^*(\omega)g^*(\nu)}{(E-\omega-\nu+i\epsilon)}$$
$$\left\{\int d\lambda \frac{|\sigma_{\lambda,B}|^2}{(E-\lambda-\omega+i\epsilon)\alpha(E-\lambda)} + \int d\mu \frac{|\sigma_{\mu,D}|^2}{(E-\mu-\nu+i\epsilon)\beta(E-\mu)} + \frac{|Z_B|^2}{\alpha(E-M_B)(E-M_B-\omega+i\epsilon)} + \frac{|Z_D|^2}{\beta(E-M_D)(E-M_D-\nu+i\epsilon)}\right\}, \tag{2.38a}$$

$$D^C(E,n,\nu) = \frac{f(n)}{\alpha(n)}\rho_{n,D}(\omega) - K_C(E,n)g^*(\nu)$$
$$\left\{\int d\mu \frac{|\sigma_{\mu,D}|^2}{(E-\mu-\nu+i\epsilon)\beta(E-\mu)} + \frac{|Z_D|^2}{\beta(E-M_D)(E-M_D-\nu+i\epsilon)}\right\}, \tag{2.38b}$$

$$B^C(E,n,\omega) = \frac{g(E-n)}{\beta(E-n)}\rho_{E-n,B}(\nu) - K_C(E,n)f^*(\omega)$$
$$\left\{\int d\lambda \frac{|\sigma_{\lambda,B}|^2}{(E-\lambda-\omega+i\epsilon)\alpha(E-\lambda)} + \frac{|Z_B|^2}{\alpha(E-M_B)(E-M_B-\omega+i\epsilon)}\right\}. \tag{2.38c}$$

For the "physical" $|D\phi(\nu)\rangle\rangle$ sector in the bare basis, we get:

$$C^D(E,\omega,\nu) = \rho_D(\omega)\rho_{E-M_D,B}(\nu) - K_D(E)\frac{f^*(\omega)g^*(\nu)}{(E-\omega-\nu+i\epsilon)}$$
$$\left\{\int d\lambda \frac{|\sigma_{\lambda,B}|^2}{(E-\lambda-\omega+i\epsilon)\alpha(E-\lambda)} + \int d\mu \frac{|\sigma_{\mu,D}|^2}{(E-\mu-\nu+i\epsilon)\beta(E-\mu)} + \frac{|Z_B|^2}{\alpha(E-M_B)(E-M_B-\omega+i\epsilon)} + \frac{|Z_D|^2}{\beta(E-M_D)(E-M_D-\nu+i\epsilon)}\right\}, \tag{2.39a}$$

$$D^D(E,\nu) = Z_D\rho_{E-M_D,B}(\nu) - K_D(E)g^*(\nu)$$

$$\left\{\int d\mu\,\frac{|\sigma_{\mu,D}|^2}{(E-\mu-\nu+i\epsilon)\beta(E-\mu)}+\frac{|Z_D|^2}{\beta(E-M_D)(E-M_D-\nu+i\epsilon)}\right\}, \tag{2.39b}$$

$$B^D(E,\omega)=\rho_D(\omega)\,\sigma_{E-M_D,B}-K_D(E)\,f^*(\omega)$$
$$\left\{\int d\lambda\,\frac{|\sigma_{\lambda,B}|^2}{(E-\lambda-\omega+i\epsilon)\alpha(E-\lambda)}+\frac{|Z_B|^2}{\alpha(E-M_B)(E-M_B-\omega+i\epsilon)}\right\}. \tag{2.39c}$$

For the "physical" $|B\theta(\omega)\rangle\rangle$ sector in the bare basis, we get:

$$C^B(E,\omega,\nu)=\rho_B(\nu)\,\rho_{E-M_B,D}(\omega)-K_B(E)\,\frac{f^*(\omega)\,g^*(\nu)}{(E-\omega-\nu+i\epsilon)}$$
$$\left\{\int d\lambda\,\frac{|\sigma_{\lambda,B}|^2}{(E-\lambda-\omega+i\epsilon)\alpha(E-\lambda)}+\int d\mu\,\frac{|\sigma_{\mu,D}|^2}{(E-\mu-\nu+i\epsilon)\beta(E-\mu)}\right.$$
$$\left.+\frac{|Z_B|^2}{\alpha(E-M_B)(E-M_B-\omega+i\epsilon)}+\frac{|Z_D|^2}{\beta(E-M_D)(E-M_D-\nu+i\epsilon)}\right\}, \tag{2.40a}$$

$$D^B(E,\nu)=\rho_B(\nu)\,\sigma_{E-M_B,D}-K_B(E)\,g^*(\nu)$$
$$\left\{\int d\mu\,\frac{|\sigma_{\mu,D}|^2}{(E-\mu-\nu+i\epsilon)\beta(E-\mu)}+\frac{|Z_D|^2}{\beta(E-M_D)(E-M_D-\nu+i\epsilon)}\right\}, \tag{2.40b}$$

$$B^B(E,\omega)=Z_B\rho_{E-M_B,D}(\omega)-K_B(E)\,f^*(\omega)$$
$$\left\{\int d\lambda\,\frac{|\sigma_{\lambda,B}|^2}{(E-\lambda-\omega+i\epsilon)\alpha(E-\lambda)}+\frac{|Z_B|^2}{\alpha(E-M_B)(E-M_B-\omega+i\epsilon)}\right\}. \tag{2.40c}$$

Finally, for the discrete states, we get:

$$C^A(M_A,\omega,\nu)=-K_A(M_A)\,\frac{f^*(\omega)\,g^*(\nu)}{M_A-\omega-\nu}$$
$$\left\{\int d\lambda\,\frac{|\sigma_{\lambda,B}|^2}{(M_A-\lambda-\omega)\alpha(M_A-\lambda)}+\int d\mu\,\frac{|\sigma_{\mu,D}|^2}{(M_A-\mu-\nu)\beta(M_A-\mu)}\right.$$
$$+\frac{|Z_B|^2}{\alpha(M_A-M_B)(M_A-M_B-\omega+i\epsilon)}$$
$$\left.+\frac{|Z_D|^2}{\beta(M_A-M_D)(M_A-M_D-\nu+i\epsilon)}\right\}, \tag{2.41a}$$

$$D^A(M_A,\nu) = -K_A(M_A)\,g^*(\nu)$$
$$\left\{\int d\mu \frac{|\sigma_{\mu,D}|^2}{(M_A-\mu-\nu)\beta(M_A-\mu)} + \frac{|Z_D|^2}{\beta(M_A-M_D)(M_A-M_D-\nu+i\epsilon)}\right\}, \tag{2.41b}$$

$$B^A(M_A,\omega) = -K_A(M_A)\,f^*(\omega)$$
$$\left\{\int d\lambda \frac{|\sigma_{\lambda,B}|^2}{(M_A-\lambda-\omega)\alpha(M_A-\lambda)} + \frac{|Z_B|^2}{\alpha(M_A-M_B)(M_A-M_B-\omega+i\epsilon)}\right\}. \tag{2.41c}$$

We now verify that Eqs. (2.38), (2.39), (2.40), and (2.41) are each solutions of our model. To do this, we explicitly write down the analogues of Eqs. (2.22) in the bare basis, plug in each set of solutions in turn, and verify that the equations are satisfied. A straightforward analysis shows that the following equations must be satisfied in the bare basis (we have written them for the "physical" $|C\theta(\omega)\phi(\nu)\rangle\rangle$ sector, *i.e.* with the variable n—for the other sectors the variable n is, of course, missing):

$$(E-\omega-\nu)C(E,n,\omega,\nu) = g^*(\nu)\,B(E,n,\omega) + f^*(\omega)\,D(E,n,\nu)\,, \tag{2.42a}$$

$$(E-m_B-\omega)\,B(E,n,\omega) = \int d\nu\, g(\nu)\,C(E,n,\omega,\nu)\,, \tag{2.42b}$$

$$(E-m_D-\nu)\,D(E,n,\nu) = \int d\omega\, f(\omega)\,C(E,n,\omega,\nu)\,. \tag{2.42c}$$

Putting each of Eqs. (2.38), (2.39), (2.40), and (2.41) in turn into Eqs. (2.42), or their equivalents for the other sectors, and using Eq. (B.14), we find that each of these sets of solutions satisfies the equations. Incidentally, a glance at Eqs. (2.42) immediately shows why we could not have solved the problem directly using them rather than the somewhat convoluted method we went through: the integral equations are not separable, and are quite intractable.

Chapter 3.
Properties of the Model and its Solutions

Orthonormality and Completeness

We now proceed to verify orthonormality and completeness of the solutions Eqs. (2.38), (2.39), (2.40), and (2.41). We start by verifying orthonormality for the diagonal components beginning with the scalar product $(\Psi^{C^\dagger}(E', n', \omega, \nu), \Psi^C(E, n, \omega, \nu))$, which is given by

$$\begin{aligned}\int d\omega\, d\nu\, \Psi^{C^\dagger}(E', n', \omega, \nu)\Psi^C(E, n, \omega, \nu)& \\ &= \int d\omega\, d\nu\, C^{C^*}(E', n', \omega, \nu)C^C(E, n, \omega, \nu) \\ &+ \int d\omega\, B^{C^*}(E', n', \omega)B^C(E, n, \omega) \\ &+ \int d\nu\, D^{C^*}(E', n', \nu)D^C(E, n, \nu). \end{aligned} \tag{3.1}$$

We now use Eqs. (2.37) to write this as

$$\begin{aligned}\int d\omega\, d\nu\, \Psi^{C^\dagger}(E', n', \omega, \nu)\Psi^C(E, n, \omega, \nu)& \\ = \int d\omega\, d\nu \Bigg[\int d\lambda'\, \rho^*_{\lambda', B}(\nu) b^{C^*}(E', n', \lambda', \omega) &+ \rho^*_B(\nu) b_F^{C^*}(E', n', M_B, \omega)\Bigg] \\ \Bigg[\int d\lambda\, \rho_{\lambda, B}(\nu) b^C(E, n, \lambda, \omega) &+ \rho_B(\nu) b_F^C(E, n, M_B, \omega)\Bigg] \\ + \int d\omega \Bigg[\int d\lambda'\, \sigma^*_{\lambda', B} b^{C^*}(E', n', \lambda', \omega) &+ Z^*_B b_F^{C^*}(E', n', M_B, \omega)\Bigg] \end{aligned}$$

$$\left[\int d\lambda\, \sigma_{\lambda,B} b^C(E,n,\lambda,\omega) + Z_B b_F^C(E,n,M_B,\omega)\right]$$
$$+ \int d\nu\, D^{C^*}(E',n',\nu) D^C(E,n,\nu). \tag{3.2}$$

We then do the integrals over λ and λ' to find

$$\begin{aligned}\int d\omega\, d\nu\, \Psi^{C\dagger}(E',n',\omega,\nu)\Psi^C(E,n,\omega,\nu) \\ = \int d\lambda\, d\omega\, b^{C^*}(E',n',\lambda,\omega) b^C(E,n,\lambda,\omega) \\ + \int d\omega\, b_F^{C^*}(E',n',M_B,\omega) b_F^C(E,n,M_B,\omega) \\ + \int d\nu\, D^{C^*}(E',n',\nu) D^C(E,n,\nu).\end{aligned} \tag{3.3}$$

Defining

$$\begin{aligned}L_1(E',n') &\equiv \frac{f^*(n')\, g^*(E'-n')}{\alpha^*(n')\, \beta^*(E'-n')}, \\ L_2(E,n) &\equiv \frac{f(n)\, g(E-n)}{\alpha(n)\, \beta(E-n)},\end{aligned} \tag{3.4}$$

we find that the sum of the first two integrals is

$$\begin{aligned}&\delta(E-E')\,\delta(n-n') - \delta(E'-n'-E+n)\,\frac{f^*(n')\, f(n)}{\alpha^*(n')\,\alpha(n)} \\ &+ L_1(E',n')\, L_2(E,n) \left\{ \frac{1}{\gamma^*(E')\,\alpha^*(E'-E+n)} + \frac{1}{\gamma(E)\,\alpha(E-E'+n')} \right\} \\ &+ \frac{L_1(E',n')\, L_2(E,n)}{\gamma^*(E')\,\gamma(E)} \left\{ \frac{-|Z_B|^2}{\alpha^*(E'-M_B)\,\alpha(E-M_B)} \right. \\ &\qquad \left. - \int d\lambda\, \frac{|\sigma_{\lambda,B}|^2}{\alpha^*(E'-\lambda)\,\alpha(E-\lambda)} \right\},\end{aligned} \tag{3.5}$$

while the third integral gives

$$\begin{aligned}&\delta(E'-n'-E+n)\,\frac{f^*(n')\, f(n)}{\alpha^*(n')\,\alpha(n)} \\ &- L_1(E',n')\, L_2(E,n) \left\{ \frac{1}{\gamma^*(E')\,\alpha^*(E'-E+n)} + \frac{1}{\gamma(E)\,\alpha(E-E'+n')} \right\}\end{aligned}$$

$$+\frac{L_1(E',n')\,L_2(E,n)}{\gamma^*(E')\,\gamma(E)}$$
$$\left\{\frac{|Z_D|^2}{\beta(E-M_D)\,\alpha^*(E'-E+M_D)}+\frac{|Z_D|^2}{\beta^*(E'-M_D)\,\alpha(E-E'+M_D)}\right.$$
$$+\int d\mu'\,\frac{|\sigma_{\mu',D}|^2}{\beta^*(E'-\mu')\,\alpha(E-E'+\mu')}$$
$$\left.+\int d\mu\frac{|\sigma_{\mu,D}|^2}{\beta(E-\mu)\,\alpha^*(E'-E-\mu)}\right\}. \tag{3.6}$$

Adding Eqs. (3.5) and (3.6) together, and doing the integrals by combining them into a single contour integral (which evaluates simply to its residues), we find that the only term left is $\delta(E'-E)\delta(n'-n)$, which is just as required.

We can similarly show that

$$\begin{aligned}\int d\omega\,d\nu\,\Psi^{D\dagger}(E',\omega,\nu)\Psi^{D}(E,\omega,\nu)&\\ =\int d\lambda\,d\omega\,b^{D^*}(E',\lambda,\omega)b^{D}(E,\lambda,\omega)&\\ +\int d\omega\,b_F^{D^*}(E',M_B,\omega)b_F^{D}(E,M_B,\omega)&\\ +\int d\nu\,D^{D^*}(E',\nu)D^{D}(E,\nu)&\\ =\delta\left(E'-E\right),&\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\int d\omega\,d\nu\,\Psi^{B\dagger}(E',\omega,\nu)\Psi^{B}(E,\omega,\nu)&\\ =\int d\lambda\,d\omega\,b^{B^*}(E',\lambda,\omega)b^{B}(E,\lambda,\omega)&\\ +\int d\omega\,b_F^{B^*}(E',M_B,\omega)b_F^{B}(E,M_B,\omega)&\\ +\int d\nu\,D^{B^*}(E',\nu)D^{B}(E,\nu)&\\ =\delta\left(E'-E\right).&\end{aligned} \tag{3.8}$$

Finally,

$$\Psi^{A\dagger}(M_A,\omega,\nu)\Psi^{A}(M_A,\omega,\nu)=1. \tag{3.9}$$

Now we move to the off-diagonal elements. For

$$\begin{aligned}\int d\omega\, d\nu\, \Psi^{C\dagger}(E', n', \omega, \nu)\Psi^{D}(E, \omega, \nu) \\ &= \int d\lambda\, d\omega\, b^{C^*}(E', n', \lambda, \omega) b^{D}(E, \lambda, \omega) \\ &+ \int d\omega\, b_F^{C^*}(E', n', M_B, \omega) b_F^{D}(E, M_B, \omega) \\ &+ \int d\nu\, D^{C^*}(E', n', \nu) D^{D}(E, \nu),\end{aligned} \tag{3.10}$$

we find that the third integral exactly cancels the sum of the first two, giving us 0. We can similarly show that

$$\int d\omega\, d\nu\, \Psi^{C\dagger}(E', n', \omega, \nu)\Psi^{B}(E, \omega, \nu) = 0, \tag{3.11}$$

$$\int d\omega\, d\nu\, \Psi^{C\dagger}(E', n', \omega, \nu)\Psi^{A}(M_A, \omega, \nu) = 0, \tag{3.12}$$

$$\int d\omega\, d\nu\, \Psi^{D\dagger}(E', \omega, \nu)\Psi^{B}(E, \omega, \nu) = 0, \tag{3.13}$$

$$\int d\omega\, d\nu\, \Psi^{D\dagger}(E', \omega, \nu)\Psi^{A}(M_A, \omega, \nu) = 0, \tag{3.14}$$

$$\int d\omega\, d\nu\, \Psi^{B\dagger}(E', \omega, \nu)\Psi^{A}(M_A, \omega, \nu) = 0. \tag{3.15}$$

Therefore, the set of solutions we found, Eqs. (2.38), (2.39), (2.40), and (2.41) are orthonormal.

We now move to completeness. We wish to show that

$$\begin{aligned}&\int dE\, dn\, \Psi^{C}(E, n, \omega, \nu)\Psi^{C\dagger}(E, n, \omega', \nu') + \int dE\, \Psi^{D}(E, \omega, \nu)\Psi^{D\dagger}(E, \omega', \nu') \\ &+ \int dE\, \Psi^{B}(E, \omega, \nu)\Psi^{B\dagger}(E, \omega', \nu') + \Psi^{A}(M_A, \omega, \nu)\Psi^{A\dagger}(M_A, \omega', \nu') \\ &= \begin{pmatrix} \delta(\nu-\nu')\,\delta(\omega-\omega') & 0 & 0 \\ 0 & \delta(\nu-\nu') & 0 \\ 0 & 0 & \delta(\omega-\omega') \end{pmatrix}.\end{aligned} \tag{3.16}$$

Let us start with the diagonal elements. The $(1,1)$ element of the matrix is

$$\int dE\, dn\, C^{C}(E, n, \omega, \nu) C^{C^*}(E, n, \omega', \nu') + \int dE\, C^{D}(E, \omega, \nu) C^{D^*}(E, \omega', \nu')$$

$$+\int dE\, C^B(E,\omega,\nu)C^{B^*}(E,\omega',\nu') + C^A(M_A,\omega,\nu)C^{A^*}(M_A,\omega',\nu'). \quad (3.17)$$

These integrals are most easily done in the following manner. The first term can be rewritten, using Eqs. (2.37), as

$$\int dEdn\, C^C(E,n,\omega,\nu)C^{C^*}(E,n,\omega',\nu')$$
$$\left\{\int d\lambda'\, \rho^*_{\lambda',B}(\nu')b^{C^*}(E,n,\lambda',\omega') + \rho^*_B(\nu')b^{C^*}_F(E,n,M_B,\omega')\right\}$$
$$= \int dE\, dn \left\{\int d\lambda\, \rho_{\lambda,B}(\nu)b^C(E,n,\lambda,\omega) + \rho_B(\nu)b^C_F(E,n,M_B,\omega)\right\}. \quad (3.18)$$

One then rewrites subsequent terms in Eq. (3.17) in a similar fashion as Eq. (3.18). Since the integrals are exceedingly tedious, we describe how they are done, and leave it to the interested reader to verify our results. The integrals over n are done with the help of Eq. (B.15). Then, the integrals over E are done by converting them into contour integrals. When all the contour integrals are combined it is found that they add together into one large contour integral (plus the non-contributing circle at infinity), which evaluates simply to its residues. These residues exactly cancel the other pieces in the expression, leaving over one or more delta functions for the diagonal terms, and nothing for the off-diagonal ones. For convenience, the branch cuts and poles of the function $\frac{1}{\gamma(z)}$, where z is a complex integration variable in the contour integral, are shown in Fig. B.3.

One finds that the $(1,1)$ term is $\delta(\omega-\omega')\delta(\nu-\nu')$. In doing this, one has to fix $K_A(M_A) = \sqrt{\frac{d\gamma(E)}{dE}|_{E=M_A}}$, which fixes the unknown normalization constant in Eqs. (2.34). One can similarly show that the $(2,2)$ and the $(3,3)$ terms are $\delta(\nu-\nu')$ and $\delta(\omega-\omega')$, respectively.

For the off-diagonal terms, one proceeds similarly and finds that they are all zero. Thus, our set of solutions, namely, Eqs. (2.38), (2.39), (2.40), and (2.41) is a complete orthonormal set of solutions of our model in this sector.

The Möller Matrix and the Comparison Hamiltonian

The matrix (with continuous eigenvalues) of the eigenfunctions, including any discrete solutions, gives us the generalized Möller Matrix by virtue of the results already demonstrated on orthonormality and completeness [17]. It is given by

$$\Omega(E,n,\omega,\nu) = \left(\Psi^C(E,n,\omega,\nu), \Psi^D(E,\omega,\nu), \Psi^B(E,\omega,\nu), \Psi^A(M_A,\omega,\nu)\right) \tag{3.19}$$

with components

$$\begin{pmatrix} C^C(E,n,\omega,\nu) & C^D(E,\omega,\nu) & C^B(E,\omega,\nu) & C^A(M_A,\omega,\nu) \\ D^C(E,n,\nu) & D^D(E,\nu) & D^B(E,\nu) & D^A(M_A,\nu) \\ B^C(E,n,\omega) & B^D(E,\omega) & B^B(E,\omega) & B^A(M_A,\omega) \end{pmatrix}. \tag{3.20}$$

It has the properties of being unitary

$$\Omega\Omega^\dagger = 1, \tag{3.21a}$$

$$\Omega^\dagger\Omega = 1, \tag{3.21b}$$

and of diagonalizing the full Hamiltonian, H,

$$H\Omega = \Omega H_C, \tag{3.22a}$$

$$\Omega^\dagger H\Omega = H_C, \tag{3.22b}$$

where H_C is called the comparison Hamiltonian. It can be calculated in the following manner. First, we use the eigenvalue equations to write

$$\begin{aligned} &H\Omega(E,n,\omega,\nu) \\ &\quad = \left(E\Psi^C(E,n,\omega,\nu), E\Psi^D(E,\omega,\nu), E\Psi^B(E,\omega,\nu), M_A\Psi^A(M_A,\omega,\nu)\right), \end{aligned} \tag{3.23}$$

and then act on Eq. (3.23) with $\Omega^\dagger$ from the left, and make use of the orthonormality relations to get

$$\Omega^\dagger H\Omega = \begin{pmatrix} E\delta(E-E')\delta(n-n') & 0 & 0 & 0 \\ 0 & E\delta(E-E') & 0 & 0 \\ 0 & 0 & E\delta(E-E') & 0 \\ 0 & 0 & 0 & M_A \end{pmatrix}$$

$$= H_C. \tag{3.24}$$

To compare H_C with the free Hamiltonian, H_0, we rewrite H_C, putting $E = n + \tau$ for the $(1,1)$ element, $E = M_D + \tau$ for the $(2,2)$ element, and $E = M_B + \tau$ for the $(3,3)$ element, and similarly for E'. Thus, H_C becomes

$$\begin{pmatrix} (n+\tau)\,\delta(\tau-\tau')\,\delta(n-n') & 0 & 0 & 0 \\ 0 & (M_D+\tau)\,\delta(\tau-\tau') & 0 & 0 \\ 0 & 0 & (M_B+\tau)\,\delta(\tau-\tau') & 0 \\ 0 & 0 & 0 & M_A \end{pmatrix}. \tag{3.25}$$

The free Hamiltonian, H_0, is

$$\begin{pmatrix} (\omega+\nu)\,\delta(\omega-\omega')\,\delta(\nu-\nu') & 0 & 0 \\ 0 & (m_D+\nu)\,\delta(\nu-\nu') & 0 \\ 0 & 0 & (m_B+\omega)\,\delta(\omega-\omega') \end{pmatrix}. \tag{3.26}$$

Comparing H_C and H_0, we see that we can identify H_C with H_0 if we include *both* mass and wave-function renormalization terms in the interaction, and ignore the discrete M_A state in H_C. The mass renormalization means that we must add a quantity Δ to H_0, where Δ is

$$\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (M_D-m_D)\,\delta(\nu-\nu') & 0 \\ 0 & 0 & (M_B-m_B)\,\delta(\omega-\omega') \end{pmatrix}. \tag{3.27}$$

The structure of our solutions, Eqs. (2.38), (2.39), and (2.40) immediately tells us that we must have a wave function (and consequent coupling constant) renormalization.

Thus, the fields B, C, D, θ, and ϕ have the wave function renormalizations

$$B \to \sqrt{\beta'}B = \frac{1}{Z_B}B, \tag{3.28a}$$

$$D \to \sqrt{\alpha'}D = \frac{1}{Z_D}D, \tag{3.28b}$$

$$C \to C, \tag{3.28c}$$

$$\theta \to \theta, \tag{3.28d}$$

$$\phi \to \phi. \tag{3.28e}$$

Because there are no proper vertex corrections, the coupling constant renormalizations reflect the wave function renormalizations [17]

$$f(\omega) \rightarrow Z_D f(\omega), \tag{3.29}$$

$$g(\nu) \rightarrow Z_B g(\nu). \tag{3.30}$$

Furthermore, as there are no divergences in this problem, the coupling constant and wave function renormalizations are inessential, and the mass renormalization making $H_0 + \Delta$ identifiable with H_C is essential only in this sector. These renormalizations are sufficient for higher sectors as well. The only change in the higher sectors is due to the mass renormalizations which alter the continuum thresholds from m_D and m_B to M_D and M_B, respectively, but leave everything else unaffected.

Notice that while H_C and H_0 have the same structure (as long as α and β both have zeros, and γ does not), they have different spectra. Only the double continuum $0 < n < E < \infty$ is coextensive; the $D\phi$ and $B\theta$ continua are renormalized downwards from m_D to M_D and from m_B to M_B, respectively. Notice also that, contrary to conventional wisdom [1,3,11], the Möller matrix intertwines the full Hamiltonian, H, with H_C, *not* with H_0. However, H_C and H do have the same spectrum.

In addition, if we take the unitary transformation of H_C in reverse, we can convert the comparison Hamiltonian to the full Hamiltonian

$$\Omega H_C \Omega^\dagger = H, \tag{3.31}$$

and just as in the Cascade Model of [17], we find that the notion of an interaction is basis dependent.

The S Matrix

We have obtained one set of solutions to our problem, namely, Eqs. (2.38), (2.39), (2.40), and (2.41). We can, of course, obtain another set in which the reciprocals of the singular operators of the form $E-\omega-\nu+i\epsilon$ (which turn out to be the "in" states) in Eqs. (2.38), (2.39), and (2.40) are changed to $E-\omega-\nu-i\epsilon$ (which turn out to be the "out" states), while Eqs. (2.41) remain unchanged. Let us denote these solutions, and quantities associated with them, with a prime. This new set also furnishes a Möller matrix,

$$\Omega' = \left(\Psi^{C'}(E,n,\omega,\nu), \Psi^{D'}(E,\omega,\nu), \Psi^{B'}(E,\omega,\nu), \Psi^{A}(M_A,\omega,\nu)\right), \quad (3.32)$$

which satisfies the same properties as the original Möller matrix, that of unitarity

$$\Omega'\Omega'^{\dagger} = \mathbf{1}, \quad (3.33a)$$

$$\Omega'^{\dagger}\Omega' = \mathbf{1}, \quad (3.33b)$$

and of diagonalizing H

$$H\Omega' = \Omega' H_C, \quad (3.34a)$$

$$\Omega'^{\dagger} H \Omega' = H_C. \quad (3.34b)$$

The set of states, Eqs. (2.38), (2.39), (2.40), and (2.41) are such that

$$\lim_{t\to-\infty} e^{iH_C t} e^{-iHt} \Psi^C(E,n,\omega,\nu) = \begin{pmatrix} \delta(n-\omega)\,\delta(E-\omega-\nu) \\ 0 \\ 0 \end{pmatrix}, \quad (3.35a)$$

$$\lim_{t\to-\infty} e^{iH_C t} e^{-iHt} \Psi^D(E,\omega,\nu) = \begin{pmatrix} 0 \\ Z_D\delta(E-M_D-\nu) \\ 0 \end{pmatrix}, \quad (3.35b)$$

$$\lim_{t\to-\infty} e^{iH_C t} e^{-iHt} \Psi^B(E,\omega,\nu) = \begin{pmatrix} 0 \\ 0 \\ Z_B\delta(E-M_B-\omega) \end{pmatrix}, \quad (3.35c)$$

$$\lim_{t\to-\infty} e^{iH_C t} e^{-iHt} \Psi^A(M_A,\omega,\nu) = \Psi^A(M_A,\omega,\nu), \quad (3.35d)$$

of which the first three are the plane wave ideal eigenstates of the comparison Hamiltonian. However, notice that there is the need for a wave function renormalization in $\Psi^D(E,\omega,\nu)$ and $\Psi^B(E,\omega,\nu)$, and that the threshold is renormalized in these two cases (*i.e.* $m_B \to M_B$ and $m_D \to M_D$). Clearly, these states are the "in" states in our problem. This is again analogous to the Cascade Model of [17].

For $t \to +\infty$ for these "in" states we have

$$\lim_{t\to+\infty} e^{iH_C t}e^{-iHt}\Psi^C(E,n,\omega,\nu)$$
$$= \begin{pmatrix} \delta(E-\omega-\nu)\left[\delta(n-\omega)\frac{\alpha^*(n)\beta^*(E-n)}{\alpha(n)\beta(E-n)} + \frac{2\pi i}{\gamma(E)}\frac{f(n)g(E-n)}{\alpha(n)\beta(E-n)}\frac{f^*(\omega)g^*(\nu)}{\alpha(\omega)\beta(\nu)}\right] \\ \frac{2\pi i}{\gamma(E)}\frac{f(n)g(E-n)}{\alpha(n)\beta(E-n)}|Z_D|^2\delta(E-M_D-\nu)\frac{g^*(\nu)}{\beta(\nu)} \\ \frac{2\pi i}{\gamma(E)}\frac{f(n)g(E-n)}{\alpha(n)\beta(E-n)}|Z_B|^2\delta(E-M_B-\omega)\frac{f^*(\omega)}{\alpha(\omega)} \end{pmatrix}, \tag{3.36a}$$

$$\lim_{t\to+\infty} e^{iH_C t}e^{-iHt}\Psi^D(E,\omega,\nu)$$
$$= \begin{pmatrix} 2\pi i\delta(E-\omega-\nu)\frac{Z_D}{\gamma(E)}\frac{g(E-M_D)}{\beta(E-M_D)}\frac{f^*(\omega)g^*(E-\omega)}{\alpha(\omega)\beta(E-\omega)} \\ Z_D\delta(E-M_D-\nu)\left[\frac{\beta^*(\nu)}{\beta(\nu)} + \frac{2\pi i}{\gamma(E)}|Z_D|^2\frac{|g(\nu)|^2}{\beta(\nu)\beta(\nu)}\right] \\ Z_B\delta(E-M_B-\omega)\frac{2\pi i}{\gamma(E)}|Z_B|^2\frac{|f(\omega)|^2}{\alpha(\omega)\alpha(\omega)} \end{pmatrix}, \tag{3.36b}$$

$$\lim_{t\to+\infty} e^{iH_C t}e^{-iHt}\Psi^B(E,\omega,\nu)$$
$$= \begin{pmatrix} 2\pi i\delta(E-\omega-\nu)\frac{Z_B}{\gamma(E)}\frac{f(E-M_B)}{\alpha(E-M_B)}\frac{f^*(\omega)g^*(E-\omega)}{\alpha(\omega)\beta(E-\omega)} \\ Z_D\delta(E-M_D-\nu)\frac{2\pi i}{\gamma(E)}|Z_D|^2\frac{|g(\nu)|^2}{\beta(\nu)\beta(\nu)} \\ Z_B\delta(E-M_B-\omega)\left[\frac{\alpha^*(\omega)}{\alpha(\omega)} + \frac{2\pi i}{\gamma(E)}|Z_B|^2\frac{|f(\omega)|^2}{\alpha(\omega)\alpha(\omega)}\right] \end{pmatrix}, \tag{3.36c}$$

$$\lim_{t\to+\infty} e^{iH_C t}e^{-iHt}\Psi^A(M_A,\omega,\nu) = \Psi^A(M_A,\omega,\nu). \tag{3.36d}$$

(The limits in Eqs. (3.35) and Eqs. (3.36) are understood for multiplication by smooth functions of ω or ν or both, as the case may be).

The "out" states behave in an analogous but opposite fashion to the "in" states. They behave simply for $t \to +\infty$, but have a complicated structure as $t \to -\infty$. Furthermore, the "in" states at $t \to -\infty$ and the "out" states at $t \to +\infty$ are identical. Therefore, we can define an S-matrix, and can compute it in one of several ways. For example, we can compute it using

$$\Psi_{\text{scattered}} = \lim_{t\to\infty} \left(\Psi(t) - \Psi(-t)\right), \tag{3.37}$$

or we can take the scalar product of the "in" and "out" states

$$(\Psi', \Psi) = S. \tag{3.38}$$

Both methods, of course, give the same answer.

The method of the inner products is cleaner and more aesthetically satisfying so we shall follow it for the calculation. The results are easily checked by doing the calculation by the other methods.

Schematically, the S-matrix looks like (the "+" subscript means an "in" state and the "−" subscript means an "out" state)

$$S = \begin{pmatrix} -\langle\langle C\theta\phi|C\theta\phi\rangle\rangle_+ & -\langle\langle C\theta\phi|D\phi\rangle\rangle_+ & -\langle\langle C\theta\phi|B\theta\rangle\rangle_+ & -\langle\langle C\theta\phi|M_A\rangle\rangle \\ -\langle\langle D\phi|C\theta\phi\rangle\rangle_+ & -\langle\langle D\phi|D\phi\rangle\rangle_+ & -\langle\langle D\phi|B\theta\rangle\rangle_+ & -\langle\langle D\phi|M_A\rangle\rangle \\ -\langle\langle B\theta|C\theta\phi\rangle\rangle_+ & -\langle\langle B\theta|D\phi\rangle\rangle_+ & -\langle\langle B\theta|B\theta\rangle\rangle_+ & -\langle\langle B\theta|M_A\rangle\rangle \\ \langle\langle M_A|C\theta\phi\rangle\rangle_+ & \langle\langle M_A|D\phi\rangle\rangle_+ & \langle\langle M_A|B\theta\rangle\rangle_+ & \langle\langle M_A|M_A\rangle\rangle \end{pmatrix}. \tag{3.39}$$

Let us start with the $(1,1)$ component of S. We wish to calculate

$$\begin{aligned} {}_C\langle\langle E', n', in|E, n, out\rangle\rangle_C \\ &= \int d\omega\, d\nu\, C^{C'^*}(E', n', \omega, \nu) C^C(E, n, \omega, \nu) \\ &+ \int d\omega\, B^{C'^*}(E', n', \omega) B^C(E, n, \omega) \\ &+ \int d\nu\, D^{C'^*}(E', n', \nu) D^C(E, n, \nu). \end{aligned} \tag{3.40}$$

We rewrite Eq. (3.40) in terms of the lower sector physical states using Eqs. (2.37) to get

$$\int d\omega\, d\nu \left[\int d\lambda'\, \rho'^*_{\lambda',B}(\nu) b^{C'^*}(E', n', \lambda', \omega) + \rho'^*_B(\nu) b^{C'}{}_F{}^*(E', n', M_B, \omega)\right]$$

$$
\begin{aligned}
&\left[\int d\lambda\, \rho_{\lambda,B}(\nu) b^{C}(E,n,\lambda,\omega) + \rho_B(\nu) b^{C}_{F}(E,n,M_B,\omega)\right] \\
&+ \int d\omega \left[\int d\lambda'\, \sigma'_{\lambda',B} b^{C'^{*}}(E',n',\lambda',\omega) + Z^{*}_{B} {b^{C'}_{F}}^{*}(E',n',M_B,\omega)\right] \\
&\qquad \left[\int d\lambda\, \sigma_{\lambda,B} b^{C}(E,n,\lambda,\omega) + Z_B b^{C}_{F}(E,n,M_B,\omega)\right] \\
&+ \int d\nu\, D^{C'^{*}}(E',n',\nu) D^{C}(E,n,\nu).
\end{aligned} \tag{3.41}
$$

Doing the integrals over λ in Eq. (3.41) we get

$$
\begin{aligned}
&\int d\lambda\, d\lambda' \frac{\beta^{*}(\lambda)}{\beta(\lambda)} \delta(\lambda - \lambda') \int d\omega\, b^{C'^{*}}(E',n',\lambda',\omega) b^{C}(E,n,\lambda,\omega) \\
&+ \int d\omega\, {b^{C'}_{F}}^{*}(E',n',M_B,\omega) b^{C}_{F}(E,n,M_B,\omega) \\
&+ \int d\nu\, D^{C'^{*}}(E',n',\nu) D^{C}(E,n,\nu).
\end{aligned} \tag{3.42}
$$

The sum of the first and second integrals gives

$$
\begin{aligned}
&\delta(E-E') \left\{ \delta(n-n') \frac{\beta^{*}(E-n)\alpha^{*}(n)}{\beta(E-n)\alpha(n)} \right. \\
&\qquad\qquad \left. + \frac{2\pi i}{\gamma(E)} \frac{g(E-n) f(n)}{\beta(E-n)\alpha(n)} \frac{g^{*}(E-n') f^{*}(n')}{\beta(E-n')\alpha(n')} \right\} \\
&- \delta(E'-n'-E+n) \frac{\beta^{*}(E-n) f^{*}(n') f(n)}{\beta(E-n)\alpha(n')\alpha(n)} \\
&+ \frac{g^{*}(E'-n') f^{*}(n') g(E-n) f(n)}{\beta(E'-n')\alpha(n')\beta(E-n)\alpha(n)} \\
&\qquad\qquad \left\{ \frac{1}{\gamma(E)\alpha(E-E'+n')} + \frac{1}{\gamma(E')\alpha(E'-E+n)} \right\} \\
&+ \frac{1}{\gamma(E')\gamma(E)} \frac{g^{*}(E'-n') f^{*}(n') g(E-n) f(n)}{\beta(E'-n')\alpha(n')\beta(E-n)\alpha(n)} \\
&\qquad \left\{ - \int d\lambda\, |\sigma_{\lambda,B}|^{2} \frac{1}{\alpha(E'-\lambda)\alpha(E-\lambda)} - \frac{|Z_B|^{2}}{\alpha(E'-M_B)\alpha(E-M_B)} \right\},
\end{aligned} \tag{3.43}
$$

and the third integral gives

$$
\delta(E'-n'-E+n) \frac{\beta^{*}(E-n) f^{*}(n') f(n)}{\beta(E-n)\alpha(n')\alpha(n)}
$$

$$
- \frac{g^*(E'-n')\, f^*(n')\, g(E-n)\, f(n)}{\beta(E'-n')\,\alpha(n')\,\beta(E-n)\,\alpha(n)}
$$

$$
\left\{ \frac{1}{\gamma(E)\,\alpha(E-E'+n')} + \frac{1}{\gamma(E')\,\alpha(E'-E+n)} \right\}
$$

$$
+ \frac{1}{\gamma(E')\,\gamma(E)} \frac{g^*(E'-n')\, f^*(n')\, g(E-n)\, f(n)}{\beta(E'-n')\,\alpha(n')\,\beta(E-n)\,\alpha(n)}
$$

$$
\left\{ \frac{1}{2} \int d\mu\, |\sigma_{\mu,D}|^2 \frac{1}{\beta(E-\mu)} \left(\frac{1}{\alpha^*(E'-E+\mu)} + \frac{1}{\alpha(E'-E+\mu)} \right) \right.
$$

$$
+ \frac{1}{2} \int d\mu'\, |\sigma_{\mu',D}|^2 \frac{1}{\beta(E'-\mu')} \left(\frac{1}{\alpha^*(E-E'+\mu')} + \frac{1}{\alpha(E-E'+\mu')} \right)
$$

$$
+ \frac{|Z_D|^2}{2\beta(E-M_D)} \left(\frac{1}{\alpha^*(E'-E+M_D)} + \frac{1}{\alpha(E'-E+M_D)} \right)
$$

$$
\left. + \frac{|Z_D|^2}{2\beta(E'-M_D)} \left(\frac{1}{\alpha^*(E-E'+M_D)} + \frac{1}{\alpha(E-E'+M_D)} \right) \right\}. \quad (3.44)
$$

Adding Eqs. (3.43) and (3.44), and converting the sum of the integrals to contour integrals (which evaluate to their residues and cancel the other terms with them inside the curly brackets), we are left with

$$
{}_C\langle\langle E', n', in | E, n, out \rangle\rangle_C = \delta(E-E') \left\{ \delta(n-n') \frac{\beta^*(E-n)\,\alpha^*(n)}{\beta(E-n)\,\alpha(n)} \right.
$$

$$
\left. + \frac{2\pi i}{\gamma(E)} \frac{g(E-n)\, f(n)}{\beta(E-n)\,\alpha(n)} \frac{g^*(E-n')\, f^*(n')}{\beta(E-n')\,\alpha(n')} \right\}. \quad (3.45)
$$

In a similar fashion, we can do all the other S-matrix elements. They are

$$
{}_D\langle\langle E', out | E, in \rangle\rangle_D = \delta(E-E') \left\{ \frac{\beta^*(E-M_D)}{\beta(E-M_D)} \right.
$$

$$
\left. + \frac{2\pi i |Z_D|^2}{\gamma(E)} \frac{|g(E-M_D)|^2}{\beta(E-M_D)\,\beta(E-M_D)} \right\}, \quad (3.46)
$$

$$
{}_B\langle\langle E', out | E, in \rangle\rangle_B = \delta(E-E') \left\{ \frac{\alpha^*(E-M_B)}{\alpha(E-M_B)} \right.
$$

$$
\left. + \frac{2\pi i |Z_B|^2}{\gamma(E)} \frac{|f(E-M_B)|^2}{\alpha(E-M_B)\,\alpha(E-M_B)} \right\}, \quad (3.47)
$$

$$ {}_A\langle\langle M_A | M_A \rangle\rangle = 1, \tag{3.48} $$

$$ {}_B\langle\langle E', out | E, in \rangle\rangle_D = 2\pi i \delta\left(E' - E\right) \frac{Z_D Z_B^*}{\gamma(E)} \frac{f^*(E - M_B)\, g(E - M_D)}{\alpha(E - M_B)\, \beta(E - M_D)}, \tag{3.49} $$

$$ {}_D\langle\langle E', out | E, in \rangle\rangle_B = 2\pi i \delta\left(E' - E\right) \frac{Z_D^* Z_B}{\gamma(E)} \frac{f(E - M_B)\, g^*(E - M_D)}{\alpha(E - M_B)\, \beta(E - M_D)}, \tag{3.50} $$

$$ {}_D\langle\langle E', out | E, n, in \rangle\rangle_C = 2\pi i \delta\left(E' - E\right) \frac{Z_D^*}{\gamma(E)} \frac{g(n)\, f(E - n)}{\alpha(n)\, \beta(E - n)} \frac{g^*(E - M_D)}{\beta(E - M_D)}, \tag{3.51} $$

$$ {}_C\langle\langle E', n', out | E, in \rangle\rangle_D = 2\pi i \delta\left(E' - E\right) \frac{Z_D}{\gamma(E)} \frac{g^*(n')\, f^*(E - n')}{\alpha(n')\, \beta(E - n')} \frac{g(E - M_D)}{\beta(E - M_D)}, \tag{3.52} $$

$$ {}_B\langle\langle E', out | E, n, in \rangle\rangle_C = 2\pi i \delta\left(E' - E\right) \frac{Z_B^*}{\gamma(E)} \frac{g(n)\, f(E - n)}{\alpha(n)\, \beta(E - n)} \frac{f^*(E - M_B)}{\alpha(E - M_B)}, \tag{3.53} $$

$$ {}_C\langle\langle E', n', out | E, in \rangle\rangle_B = 2\pi i \delta\left(E' - E\right) \frac{Z_B}{\gamma(E)} \frac{g^*(n')\, f^*(E - n')}{\alpha(n')\, \beta(E - n')} \frac{f(E - M_B)}{\alpha(E - M_B)}, \tag{3.54} $$

$$ {}_A\langle\langle M_A | E, n, in \rangle\rangle_C = 0, \tag{3.55} $$

$$ {}_C\langle\langle E', n', out | M_A \rangle\rangle_A = 0, \tag{3.56} $$

$$ {}_A\langle\langle M_A | E, in \rangle\rangle_D = 0, \tag{3.57} $$

$$ {}_D\langle\langle E', out | M_A \rangle\rangle_A = 0, \tag{3.58} $$

$$ {}_A\langle\langle M_A | E, in \rangle\rangle_B = 0, \tag{3.59} $$

$$ {}_B\langle\langle E', out | M_A \rangle\rangle_A = 0. \tag{3.60} $$

Unitarity of the S Matrix

We can almost trivially show that the S matrix that we have obtained is unitary. In equations, we wish to show that

$$SS^{\dagger}=1. \tag{3.61}$$

Let us calculate the $(1,1)$ term in $SS^{\dagger}$. It is

$$\begin{aligned}
&SS^{\dagger}_{(1,1)}\\
&\quad=\delta\left(E-E'\right)\int dn''\left[\delta\left(n-n''\right)\frac{\beta^{*}\left(E-n\right)\alpha^{*}\left(n\right)}{\beta\left(E-n\right)\alpha\left(n\right)}\right.\\
&\qquad\qquad\qquad\left.+\frac{2\pi i}{\gamma\left(E\right)}\frac{f^{*}\left(n''\right)f\left(n\right)g^{*}\left(E-n''\right)g\left(E-n\right)}{\alpha\left(n''\right)\alpha\left(n\right)\beta\left(E-n''\right)\beta\left(E-n\right)}\right]\\
&\qquad\qquad\left[\delta\left(n'-n''\right)\frac{\beta\left(E-n'\right)\alpha\left(n'\right)}{\beta^{*}\left(E-n'\right)\alpha^{*}\left(n'\right)}\right.\\
&\qquad\qquad\qquad\left.-\frac{2\pi i}{\gamma^{*}\left(E\right)}\frac{f\left(n''\right)f^{*}\left(n'\right)g\left(E-n''\right)g^{*}\left(E-n'\right)}{\alpha^{*}\left(n''\right)\alpha^{*}\left(n'\right)\beta^{*}\left(E-n''\right)\beta^{*}\left(E-n'\right)}\right]\\
&\qquad-\frac{\left(2\pi i\right)^{2}}{\left|\gamma\left(E\right)\right|^{2}}\delta\left(E-E'\right)\frac{f\left(n\right)g\left(E-n\right)}{\alpha\left(n\right)\beta\left(E-n\right)}\frac{f^{*}\left(n'\right)g^{*}\left(E-n'\right)}{\alpha^{*}\left(n'\right)\beta^{*}\left(E-n'\right)}\\
&\qquad\qquad\left[\left|Z_{D}\right|^{2}\frac{\left|g\left(E-M_{D}\right)\right|^{2}}{\left|\beta\left(E-M_{D}\right)\right|^{2}}+\left|Z_{B}\right|^{2}\frac{\left|f\left(E-M_{B}\right)\right|^{2}}{\left|\alpha\left(E-M_{B}\right)\right|^{2}}\right].
\end{aligned} \tag{3.62}$$

Doing the integral in Eq. (3.62) with the help of Eq. (B.15) we find that the result is $\delta(E-E')\delta(n-n')$, exactly as desired. The rest of the terms are done in the same way. We find

$$SS^{\dagger}_{(2,2)}=\delta\left(E-E'\right), \tag{3.63}$$

$$SS^{\dagger}_{(3,3)}=\delta\left(E-E'\right), \tag{3.64}$$

$$SS^{\dagger}_{(4,4)}=1, \tag{3.65}$$

with all other terms in $SS^{\dagger}$ being zero, as required. Thus $SS^{\dagger}=\mathbf{1}$. In the same way, we can also show that $S^{\dagger}S=\mathbf{1}$, and therefore, our S-matrix is unitary.

Eigenphases of the S Matrix

The interesting case for the S Matrix is when $E > 0$ so that all channels are open. The S Matrix must satisfy

$$S\zeta = \tau\zeta, \tag{3.66}$$

where $|\tau|^2 = 1$, for some ζ. This is equivalent to the following relations (where we ignore the discrete A channel, as it is decoupled from everything else, and suppress $\delta(E - E'))$

$$\begin{aligned}\tau - &\frac{\beta^*(E-n)\,\alpha(n)}{\beta(E-n)\,\alpha(n)}\zeta_n \\ &= \frac{2\pi i}{\gamma(E)}\frac{f(n)g(E-n)}{\alpha(n)\beta(E-n)}\left\{\int dn'\,\frac{f^*(n')g^*(E-n')}{\alpha(n')\beta(E-n')}\zeta_{n'} + Z_D^*\frac{g^*(E-M_D)}{\beta(E-M_D)}\zeta_D\right. \\ &\qquad\qquad \left. + Z_B^*\frac{f^*(E-M_B)}{\alpha(E-M_B)}\zeta_B\right\},\end{aligned} \tag{3.67a}$$

$$\begin{aligned}\tau - &\frac{\beta^*(E-M_D)}{\beta(E-M_D)}\zeta_D \\ &= \frac{2\pi i}{\gamma(E)}Z_D\frac{g(E-M_D)}{\beta(E-M_D)}\left\{\int dn'\,\frac{f^*(n')g^*(E-n')}{\alpha(n')\beta(E-n')}\zeta_{n'} + Z_D^*\frac{g^*(E-M_D)}{\beta(E-M_D)}\zeta_D\right. \\ &\qquad\qquad \left. + Z_B^*\frac{f^*(E-M_B)}{\alpha(E-M_B)}\zeta_B\right\},\end{aligned} \tag{3.67b}$$

$$\begin{aligned}\tau - &\frac{\alpha^*(E-M_B)}{\alpha(E-M_D)}\zeta_B \\ &= \frac{2\pi i}{\gamma(E)}Z_B\frac{f(E-M_B)}{\alpha(E-M_B)}\left\{\int dn'\,\frac{f^*(n')g^*(E-n')}{\alpha(n')\beta(E-n')}\zeta_{n'} + Z_D^*\frac{g^*(E-M_D)}{\beta(E-M_D)}\zeta_D\right. \\ &\qquad\qquad \left. + Z_B^*\frac{f^*(E-M_B)}{\alpha(E-M_B)}\zeta_B\right\}.\end{aligned} \tag{3.67c}$$

We now define the unimodular quantities

$$\tau(n) \equiv \frac{\beta^*(E-n)\,\alpha^*(n)}{\beta(E-n)\,\alpha(n)}, \tag{3.68a}$$

$$\tau_D \equiv \frac{\beta^*(E-M_D)}{\beta(E-M_D)}, \tag{3.68b}$$

$$\tau_B \equiv \frac{\alpha^*(E-M_B)}{\alpha(E-M_B)}. \tag{3.68c}$$

These are the basic equations. We can solve them for continuum values or for discrete values of the eigenphase shifts. Let us start with the continuum values. We invert Eq. (3.67a) and put a delta function on the right hand side along with the appropriate normalization. We then multiply both sides of the equation by $\frac{f^*(n)g^*(E-n)}{\alpha(n)\beta(E-n)}$ and integrate over n to get

$$\left\{1 - \frac{2\pi i}{\gamma(E)}\int \frac{|f(l)|^2|g(E-l)|^2}{|\alpha(l)|^2|\beta(E-l)|^2}\frac{\tau(l)}{\tau-\tau(l)+i\epsilon}\right\}\int dn'\,\frac{f^*(n')g^*(E-n')}{\alpha(n')\beta(E-n')}\zeta_{n'}$$
$$= \frac{f^*(\tau)g^*(\tau)}{\alpha(\tau)\beta(\tau)} + \frac{2\pi i}{\gamma(E)}\int \frac{|f(l)|^2|g(E-l)|^2}{|\alpha(l)|^2|\beta(E-l)|^2}\frac{\tau(l)}{\tau-\tau(l)+i\epsilon}$$
$$\left\{Z_D^*\frac{g^*(E-M_D)}{\beta(E-M_D)}\zeta_D + Z_B^*\frac{f^*(E-M_B)}{\alpha(E-M_B)}\zeta_B\right\}. \quad (3.69)$$

Defining

$$\Sigma \equiv 1 - \frac{2\pi i}{\gamma(E)}|Z_D|^2\frac{|g(E-M_D)|^2}{|\beta(E-M_D)|^2}\frac{\sigma_D}{\sigma-\sigma_D}$$
$$- \frac{2\pi i}{\gamma(E)}|Z_B|^2\frac{|f(E-M_B)|^2}{|\alpha(E-M_B)|^2}\frac{\sigma_B}{\sigma-\sigma_B}, \quad (3.70)$$

we invert Eq. (3.67b) and Eq. (3.67c) to solve for the term in the curly braces on the right hand side of Eq. (3.69), namely, $Z_D^*\frac{g^*(E-M_D)}{\beta(E-M_D)}\zeta_D + Z_B^*\frac{f^*(E-M_B)}{\alpha(E-M_B)}\zeta_B$, and find

$$Z_D^*\frac{g^*(E-M_D)}{\beta(E-M_D)}\zeta_D + Z_B^*\frac{f^*(E-M_B)}{\alpha(E-M_B)}\zeta_B$$
$$= \left(\frac{1}{\Sigma}-1\right)\int dn'\,\frac{f^*(n')g^*(E-n')}{\alpha(n')\beta(E-n')}\zeta_{n'}. \quad (3.71)$$

We now define

$$\chi(\tau) = 1 - \frac{2\pi i}{\gamma(E)\Sigma}\int dl\,\frac{|f(l)|^2|g(E-l)|^2}{|\alpha(l)|^2|\beta(E-l)|^2}\frac{\tau(l)}{\tau-\tau(l)+i\epsilon}, \quad (3.72)$$

and use this to combine Eq. (3.69) and Eq. (3.71), and find

$$\int dn'\,\frac{f^*(n')g^*(E-n')}{\alpha(n')\beta(E-n')}\zeta_{n'} + Z_D^*\frac{g^*(E-M_D)}{\beta(E-M_D)}\zeta_D + Z_B^*\frac{f^*(E-M_B)}{\alpha(E-M_B)}\zeta_B$$
$$= \frac{1}{\chi(\tau)\Sigma}\frac{f^*(\tau)g^*(\tau)}{\alpha(\tau)\beta(E-\tau)}. \quad (3.73)$$

Therefore, our continuum solutions are

$$\zeta_n = \sqrt{\tau'}\delta(\tau - \tau(n)) + \frac{2\pi i}{\gamma(E)} \frac{f(n)g(E-n)}{\alpha(n)\beta(E-n)} \frac{1}{(\tau - \tau(n) + i\epsilon)\chi(\tau)\Sigma} \frac{f^*(\tau)g^*(E-\tau)}{\alpha(\tau)\beta(E-\tau)}, \tag{3.74a}$$

$$\zeta_D = \frac{2\pi i}{\gamma(E)} Z_D \frac{g(E-M_D)}{\beta(E-M_D)} \frac{1}{(\tau - \tau_D + i\epsilon)\chi(\tau)\Sigma} \frac{f^*(\tau)g^*(E-\tau)}{\alpha(\tau)\beta(E-\tau)}, \tag{3.74b}$$

$$\zeta_B = \frac{2\pi i}{\gamma(E)} Z_B \frac{f(E-M_B)}{\alpha(E-M_B)} \frac{1}{(\tau - \tau_B + i\epsilon)\chi(\tau)\Sigma} \frac{f^*(\tau)g^*(E-\tau)}{\alpha(\tau)\beta(E-\tau)}. \tag{3.74c}$$

To investigate the spectrum of τ, we use the method of [17]. We define the following quantities, taking advantage of their being unimodular:

$$e^{2i\theta(n)} \equiv \tau(n), \tag{3.75a}$$

$$e^{2i\theta_D} \equiv \tau_D, \tag{3.75b}$$

$$e^{2i\theta_B} \equiv \tau_B, \tag{3.75c}$$

$$e^{2i\delta} \equiv \tau. \tag{3.75d}$$

We then put these definitions in Eqs. (3.74), and see that $\tau(n) = e^{2i\theta(n)}$ ranges continuously along a unit circle in the complex plane from $\theta = \theta(0)$ to $\theta = \theta(E)$.

In addition, these solutions are continuum normalized

$$\int dn\, \zeta_n(\tau' - i\epsilon)\, \zeta_n(\tau + i\epsilon) + \zeta_D(\tau' - i\epsilon)\, \zeta_D(\tau + i\epsilon) + \zeta_B(\tau' - i\epsilon)\, \zeta_B(\tau + i\epsilon) = \delta(\tau' - \tau), \tag{3.76}$$

and will be complete if there are no discrete zeros of $\chi(\tau)$. If there are, they will have to be included in the completeness identity. We now find the number of discrete zeros of $\chi(\tau)$, that is, the number of discrete eigenphase shifts of our S-Matrix.

We define

$$\tau \equiv \frac{1+ix}{1-ix}, \quad \tau(n) \equiv \frac{1+ix(n)}{1-ix(n)},$$
$$\tau_D \equiv \frac{1+ix_D}{1-ix_D}, \quad \tau_B \equiv \frac{1+ix_B}{1-ix_B},$$

put these in Eq. (3.72), and take real and imaginary parts to get

$$\begin{aligned} -\frac{1}{2}\int dl \frac{|f(l)|^2|g(E-l)|^2}{|\alpha(l)|^2|\beta(E-l)|^2}\frac{x(1+x(l))}{x-x(l)+i\epsilon} \\ = \mathrm{Im}\left(\frac{\gamma(E)}{2\pi i}\right) - \frac{1}{2}\frac{x(1+x_D)}{x-x_D}|Z_D|^2\frac{|g(E-M_D)|^2}{|\beta(E-M_D)|^2} \\ - \frac{1}{2}\frac{x(1+x_B)}{x-x_B}|Z_B|^2\frac{|f(E-M_B)|^2}{|\alpha(E-M_B)|^2}, \end{aligned} \tag{3.77a}$$

$$\begin{aligned} -\frac{1}{2}\int dl \frac{|f(l)|^2|g(E-l)|^2}{|\alpha(l)|^2|\beta(E-l)|^2} \\ = \mathrm{Re}\left(\frac{\gamma(E)}{2\pi i}\right) - \frac{1}{2}|Z_D|^2\frac{|g(E-M_D)|^2}{|\beta(E-M_D)|^2} \\ - \frac{1}{2}|Z_B|^2\frac{|f(E-M_B)|^2}{|\alpha(E-M_B)|^2}. \end{aligned} \tag{3.77b}$$

We observe that Eq. (3.77b) is an identity, by means of Eq. (B.15). To find the number of zeros of $X(\tau)$, we multiply both sides of Eq. (3.77a) by $(x - x_D)(x - x_B)$. We then find that the highest power of x appearing in Eq. (3.77a) is x^3, barring any higher powers contributed by the integral. Therefore, there are at least three discrete zeros of $X(\tau)$, and thus, at least three discrete solutions which will have to be included in the completeness identity, Eq. (3.76).

Chapter 4.
Scattering Theory

There are many different approaches to Quantum Scattering. The most familiar of these is potential scattering. Others include the LSZ formalism, the "almost local" formalism, the Lax-Phillips* formalism, and the Sudarshan approach.

The well known LSZ formalism [5], extended by Mohan [21], postulates the convergence of the matrix elements of interacting fields to the matrix elements of free fields. However, the formalism does not apply in many cases. For example, as noted by LSZ themselves, it is inapplicable to problems in which stable bound states exist. Trouble occurs when this point is forgotten, and the formalism is extended into areas where it is inapplicable. The "almost local" formalism due to Haag [1], Ruelle [8], Ekstein [9], Jauch [11], Araki [22], and others tries to be general enough to consider complicated problems [1]. Its basic idea is that it is possible to construct asymptotic in-going and out-going states as strong limits in Hilbert space, if a certain "space like asymptotic condition" is verified by the vacuum expectation values of products of field operators [8]: the so called "almost local" operators [1]. The Sudarshan approach, due to Sudarshan and collaborators [23-29], takes a very different view of scattering problems, and is quite different in spirit. The idea is that one always works with the complete set of eigenstates of the full Hamiltonian, H, properly labelled. This set, by definition, is both orthonormal and complete. The matrix made up of these eigenstates is the

* Lax-Phillips [20] theory is outside the scope of this work.

Möller matrix. This Möller matrix, again by definition, will diagonalize the full Hamiltonian giving the comparison Hamiltonian. In other words,

$$H\Omega = \Omega H_C. \tag{4.1}$$

This Möller matrix has the property that

$$\Omega\Omega^\dagger = \mathbf{1}, \quad \Omega^\dagger\Omega = \mathbf{1}. \tag{4.2}$$

It is thus isometric and unitary, as long as we ensure that the spectra of H and H_C are the same, and the spectrum multiplicity is properly preserved. The only caveat here is that not all Hamiltonians have a complete set of eigenstates. However, almost all "reasonable" Hamiltonians will have a such complete set.

This work is in the spirit of the Sudarshan approach. We shall restrict our attention to potential scattering and the "almost local" formalism as they are quite "generic"; as mentioned before, the LSZ formalism is *not applicable* to situations where stable bound states are present, such as our model.

We start with short reviews of potential scattering, for both the single and the multiple channel cases, and the "almost local" formalism of Haag and Ekstein. We then compare the results from these two formalisms with the results obtained from the Rearrangement Model.

For our purposes, it makes no difference whether we use the time dependent or time independent formalisms of scattering theory. We are concerned with the assumptions and results that obtain from the formalisms, and they remain essentially the same in both cases. We shall follow the time dependent formalism: the interested reader is referred to Newton [30] for details of the time-independent formalism.

Potential Scattering

In this section, we follow the development of Newton [30]. We wish to solve the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(t) = H\Psi(t). \tag{4.3}$$

We split H into a free Hamiltonian and an interaction Hamiltonian,

$$H = H_0 + H'. \tag{4.4}$$

We assume that this split can be carried out: we shall consider the case of rearrangement collisions later. We define four Green's functions

$$\left(i\frac{\partial}{\partial t} - H_0\right) G^{\pm}(t) = \mathbf{1}\delta(t), \tag{4.5a}$$

$$\left(i\frac{\partial}{\partial t} - H\right) \mathcal{G}^{\pm}(t) = \mathbf{1}\delta(t), \tag{4.5b}$$

with the initial conditions

$$G^{+}(t) = \mathcal{G}^{+}(t) = 0, \quad t < 0, \tag{4.6a}$$

$$G^{-}(t) = \mathcal{G}^{-}(t) = 0, \quad t > 0. \tag{4.6b}$$

G^{+} and $\mathcal{G}^{+}$ are therefore the advanced Green's functions, and G^{-} and $\mathcal{G}^{-}$ are the retarded Green's functions.

These may be solved formally yielding

$$G^{+}(t) = -ie^{-iH_0t}\theta(t), \tag{4.7a}$$

$$G^{-}(t) = ie^{-iH_0t}\theta(-t), \tag{4.7b}$$

$$\mathcal{G}^{+}(t) = -ie^{-iHt}\theta(t), \tag{4.7c}$$

$$\mathcal{G}^{-}(t) = ie^{-iHt}\theta(-t). \tag{4.7d}$$

Let $\Psi_0(t)$ be a state vector satisfying the free Schrödinger equation. The operator G^{+} can then be used to express the state vector $\Psi_0(t')$ for any time t' later than t, in terms of its value at $t' = t$,

$$\Psi_0(t') = iG^{+}(t'-t)\,\Psi_0(t). \tag{4.8}$$

$\Psi_0(t')$ then satisfies the free Schrödinger equation for $t' > t$, and $\Psi_0(t') \rightarrow \Psi_0(t)$ when $t' \rightarrow t^+$.

Therefore,

$$\lim_{t \to 0^+} G^+(t) = \lim_{t \to 0^+} \mathcal{G}^+(t) = -i\mathbf{1}, \tag{4.9a}$$

$$\lim_{t \to 0^-} G^-(t) = \lim_{t \to 0^-} \mathcal{G}^-(t) = i\mathbf{1}. \tag{4.9b}$$

Similarly, for $t' > t$ we can write

$$\Psi(t') = i\mathcal{G}^+(t'-t)\,\Psi(t), \tag{4.10}$$

and for $t' < t$ we have

$$\Psi_0(t') = -iG^-(t'-t)\,\Psi_0(t), \tag{4.11a}$$

$$\Psi(t') = -i\mathcal{G}^-(t'-t)\,\Psi(t). \tag{4.11b}$$

We now wish to define "in" and "out" states. We start by defining

$$\Psi_0(t) \equiv iG^+(t-t')\,\Psi(t'), \tag{4.12}$$

whose time development for $t > t'$ is governed by the free Hamiltonian, but which at time t_0 was equal to $\Psi(t_0)$. We now let the time, t', approach $\pm\infty$. Then, for the case of $t \rightarrow +\infty$, we have the "out" state, and for the case of $t \rightarrow -\infty$, we have the "in" state. In terms of the "in" and "out" states, the equations for $\Psi(t)$ are

$$\Psi(t) = \Psi_{\text{in}}(t) + \int_{-\infty}^{+\infty} dt'\,\mathcal{G}^+(t-t')\,H'\Psi_{\text{in}}(t'), \tag{4.13a}$$

$$= \Psi_{\text{out}}(t) + \int_{-\infty}^{+\infty} dt'\,\mathcal{G}^-(t-t')\,H'\Psi_{\text{in}}(t'). \tag{4.13b}$$

Note that these are retarded and advanced Green's functions for the whole system. These are not the same functions as those that appear in a (time ordered) Dyson series which are, instead, time ordered particle propagators.

Note also that for every state in the continuous spectrum of H_0, and only for such states, there is a corresponding state in the spectrum of H.

If we insert Eq. (4.11b) in Eq. (4.13a), we find

$$\Psi(t) = \Omega^{(+)} \Psi_{\text{in}}(t), \tag{4.14}$$

where

$$\begin{aligned} \Omega^{(+)} &= \mathbf{1} - i \int_{-\infty}^{+\infty} dt' \mathcal{G}^{+}(t - t') H' G^{-}(t' - t) \\ &= \mathbf{1} - i \int_{-\infty}^{+\infty} dt\, \mathcal{G}^{+}(-t) H' G^{-}(t) \end{aligned} \tag{4.15}$$

is called the wave operator or the Möller matrix. We can similarly find $\Omega^{(-)}$.

Because H' is hermitian, Eq. (4.15) gives us the relation

$$\Psi_{\text{in}}(t) = \Omega^{(+)\dagger} \Psi(t), \tag{4.16a}$$

and similarly

$$\Psi_{\text{out}}(t) = \Omega^{(-)\dagger} \Psi(t). \tag{4.16b}$$

Then, Eq. (4.14) and Eqs. (4.16) give us the relations

$$\Psi_{\text{in}}(t) = \Omega^{(+)\dagger} \Omega^{(+)} \Psi_{\text{in}}(t), \tag{4.17a}$$

$$\Psi_{\text{out}}(t) = \Omega^{(-)\dagger} \Omega^{(-)} \Psi_{\text{out}}(t). \tag{4.17b}$$

We now consider the possibility that the free states $\Psi_{\text{in}}(t)$ and $\Psi_{\text{out}}(t)$ span the entire Hilbert space, *i.e.* they are complete. From this assumption, we conclude that $\Omega^{(+)}$ and $\Omega^{(-)}$ are isometric,

$$\Omega^{(+)\dagger} \Omega^{(+)} = \Omega^{(-)\dagger} \Omega^{(-)} = \mathbf{1}. \tag{4.18}$$

This does not, however, mean that the Ω's are unitary: we cannot conclude from Eq. (4.17a) that Eq. (4.18) holds with its factors of Ω reversed.

Furthermore, because of the assumption that the Ψ_{in} and Ψ_{out} each form complete sets, we conclude that

$$H\Omega^{(\pm)} = \Omega^{(\pm)}H_0. \tag{4.19}$$

When there are bound states in the spectrum of H, we proceed as follows. Let $\Psi_0(E,\alpha)$ be the eigenstates of the free Hamiltonian with eigenvalue E, and α be the set of variables necessary to remove any degeneracy. Then, the completeness* of these states can be written as a resolution of the identity,

$$\mathbf{1} = \sum_\alpha \int_0^\infty dE\, \Psi_0(E,\alpha)\, \Psi_0{}^\dagger(E,\alpha). \tag{4.20}$$

We then insert this into the product $\Omega\Omega^\dagger$, to get

$$\begin{aligned}\Omega\Omega^\dagger &= \Omega \int_0^\infty dE \sum_\alpha \Psi_0(E,\alpha)\, \Psi_0{}^\dagger(E,\alpha)\, \Omega^\dagger \\ &= \int_0^\infty dE \sum_\alpha \Psi(E,\alpha)\, \Psi^\dagger(E,\alpha) \\ &= \mathbf{1} - \Lambda.\end{aligned} \tag{4.21}$$

Λ is called the unitary deficiency of Ω. From the completeness of the set of all states, bound and scattering, of H,

$$\Lambda = \sum_n \Psi_{\text{bd}}^{(n)} \Psi_{\text{bd}}^{(n)\dagger}. \tag{4.22}$$

Thus, Λ projects onto the space spanned by the bound states of H. If H has no bound states, then $\Omega^{(+)}$ and $\Omega^{(-)}$ are unitary. Both H and H_0 are hermitian; therefore, the hermitian conjugate of Eq. (4.19) gives

$$H_0\Omega^\dagger = \Omega^\dagger H. \tag{4.23}$$

* We again emphasize that we do not know whether the states of H_0 are complete, a priori. We are simply proceeding under that assumption.

We now let both sides of Eq. (4.23) act on $\Psi(E,\alpha)$ to get

$$H_0\Omega^{\dagger}\Psi(E,\alpha) = E\Omega^{\dagger}\Psi(E,\alpha), \tag{4.24}$$

which shows that if E is in the spectrum of H but not in the spectrum of H_0 then

$$\Omega^{\dagger}\Psi(E,\alpha) = 0,$$

and so

$$\Omega^{\dagger}\Lambda = 0. \tag{4.25}$$

Thus, the range of the operators $\Omega^{(\pm)}$ is not the entire Hilbert space. Instead, these operators map the whole space onto the subspace spanned by the continuum eigenstates of H. We cannot reach the subspace spanned by the bound states of H, and therefore, cannot construct an inverse operator for the whole space. The closest that we can come is to use the operators $\Omega^{(\pm)\dagger}$ which are inverses of $\Omega^{(\pm)}$ on the subspace of states spanned by the scattering states of H, and which annihilate the subspace of bound states of H.

Assuming that the asymptotic states are complete, we construct the S-matrix in the following manner. We use Eqs. (4.17b) and (4.14) to write the "out" state in terms of the "in" state,

$$\Psi_{\text{out}}(t) = \Omega^{(-)}\Omega^{(+)}\Psi_{\text{in}}(t), \tag{4.26}$$

which defines for us the S-matrix

$$S \equiv \Omega^{(-)\dagger}\Omega^{(+)}. \tag{4.27}$$

The S-matrix can be shown to be unitary and isometric. See Newton [30] for details; note, however, that S is only unitary when Ω is unitary. This point is not clear in Newton, or in the literature.

Multiple Channel Scattering

The above formalism is only adequate for simple single-channel cases. For more general scattering problems, such as rearrangement collisions, we must generalize the formalism. We shall again follow Newton [30].

We want to split up the Hamiltonian into two pieces: one piece, H_a, that is left when the two initial fragments are taken far apart, and the remaining piece, H'_a. We can then go through the same development of $\Psi(t)$ from $\Psi_{\text{in}}(t)$ as above. However, there is a difficulty that occurs for the development for the distant future. If rearrangements or break-ups can occur, then it is possible that the "channel" Hamiltonian in the future is different than the "channel" Hamiltonian in the past.

The various possibilities for an n-particle system are handled by defining a partition of them into k clusters, denoted by a_k. Given a partition a, we define H_a by allowing all distances between clusters to independently tend to infinity. Therefore, H_a will contain only interactions that are internal to clusters, but none between them. Then, H'_a is defined by the requirement that $H = H_a + H'_a$, and therefore, for any two partitions a and b, we have

$$H = H_a + H'_a = H_b + H'_b. \tag{4.28}$$

To each partition, there will correspond Green's functions given by

$$\left(i\frac{\partial}{\partial t} - H_a\right) G^{\pm}{}_a(t) = \mathbf{1}\delta(t), \tag{4.29}$$

with the same boundary conditions as Eqs. (4.6). If H_a, after removing the kinetic energy of the center of mass motion and of the centers of mass of its clusters, has at least one bound state, it is called an arrangement channel. When this condition on H_a does not hold, the channel is not of interest as an initial or final scattering state. If H_a has more than one bound state, then each of them defines a separate channel, and therefore, in each channel the clusters are in a specific bound state but moving freely relative to one another. The channel consisting of the entire n-cluster partition is the channel 0 because then $H_a = H_0$.

Now consider the space of each arrangement channel a, which we shall denote by $\mathcal{H}_a$. Then, if a has m fragments, each state in $\mathcal{H}_a$ will have m groups of bound particles. This means that unless the channel a is the entire n-fragment arrangement channel, $\mathcal{H}_a$ will not be the whole Hilbert space: the ionized eigenstates of H_a will be missing. Furthermore, as each $\mathcal{H}_a$ is defined by different channel Hamiltonians, H_a, the $\mathcal{H}_a$'s are generally not orthogonal to each other. In fact, "the complete set of basis functions is not linearly independent and, of course, not orthonormal" [32].

It will be convenient to define the orthogonal projections P_a onto the channel spaces, $\mathcal{H}_a$. In other words, we define

$$P_a^2 = P_a, \quad P_a^\dagger = P_a, \quad P_a\mathcal{H}_a = \mathcal{H}_a, \tag{4.30}$$

with the null space of P_a defined as the space spanned by the ionized eigenstates of H_a. P_a projects states from the full Hilbert space, $\mathcal{H}$, to the channel spaces, $\mathcal{H}_a$. Obviously, for the n-cluster arrangement channel, we have $P_0 = \mathbf{1}$.

We now wish to define "in" and "out" states. We first define an a state, which is a state that develops according to H_a but is in $\mathcal{H}_a$,

$$\left(i\frac{\partial}{\partial t} - H_a\right)\Psi_a(\alpha, t) = 0, \tag{4.31}$$

where the label α contains all the other information including the arrangement channel (even though including the arrangement channel in α is redundant for $\Psi_a(\alpha, t)$, it is convenient for other purposes).

We then define $\Psi^{(+)}(\alpha, t)$ as a state in $\mathcal{H}$ that develops according to H,

$$\left(i\frac{\partial}{\partial t} - H\right)\Psi^{(+)}(\alpha, t) = 0, \tag{4.32}$$

and for which there exists an a-state such that

$$\lim_{t\to-\infty}\left(\Psi_a(\alpha,t),\Psi^{(+)}(\alpha,t)\right)=1. \tag{4.33}$$

Therefore, the state $\Psi_a(\alpha,t)$ is the "in" state $\Psi_{\text{in}}(\alpha,t)$ in relation to the state $\Psi^{(+)}(\alpha,t)$. Eq. (4.33) demands that the probability of finding the system in state $\Psi_a(\alpha,t)$ in the remote past approach 1, and therefore, it is equivalent to

$$\Psi_a^{(+)} \quad \underset{t\to-\infty}{\Rightarrow} \quad \Psi_a(\alpha,t), \tag{4.34a}$$

or

$$\int dt\, iG_a^{+}(t-t')\,\Psi^{(+)}(\alpha,t') \quad \underset{t\to-\infty}{\Rightarrow} \quad \Psi_{\text{in}}(\alpha,t), \tag{4.34b}$$

with the double arrow denoting the strong limit.

Similarly,

$$\int dt'\,(-i)\,G_a^{-}(t-t')\,\Psi^{(-)}(\alpha,t') \quad \underset{t\to+\infty}{\Rightarrow} \quad \Psi_{\text{out}}(\alpha,t). \tag{4.35}$$

Exactly analogous to Eqs. (4.13), we can now write

$$\Psi^{(+)}(\alpha,t)=\Psi_{\text{in}}(\alpha,t)+\int_{-\infty}^{+\infty}dt'\,\mathcal{G}^{+}(t-t')\,H_a'\Psi_{\text{in}}(\alpha,t'), \tag{4.36a}$$

$$\Psi^{(-)}(\alpha,t)=\Psi_{\text{out}}(\alpha,t)+\int_{-\infty}^{+\infty}dt'\,\mathcal{G}^{-}(t-t')\,H_a'\Psi_{\text{out}}(\alpha,t'). \tag{4.36b}$$

We can now define the Möller matrices, and the S-matrix. The Möller matrices are defined by

$$\Psi^{\pm}(\alpha,t)=\Omega_a^{(\pm)}\Psi_a(\alpha,t), \tag{4.37}$$

with only those states Ψ_a admitted which are in $\mathcal{H}_a$. On the orthogonal complement (*i.e.* the ionized eigenstates of H_a) $\Omega^{(\pm)}$ is defined to be zero,

$$\Omega^{(\pm)}P_a=\Omega^{(\pm)}.$$

Then, on the space $\mathcal{H}_a$, using Eq. (4.36a), we find

$$\begin{aligned}\Omega_a^{(+)} &= P_a + K_a^{(+)},\\ K_a^{(+)} &= -i\int_{-\infty}^{\infty} dt\, \mathcal{G}^{+}(-t)\, H_a' G^{-}{}_a(t)\, P_a\\ &= -i\int_{-\infty}^{0} dt\, e^{iHt} H_a' e^{-iH_a t} P_a,\end{aligned}$$

and therefore,

$$\Omega_a^{(+)} = \lim_{t\to-\infty} e^{iHt} e^{-iH_a t} P_a. \tag{4.38}$$

We can similarly find, on $\mathcal{H}_a$, that

$$\Omega_a^{(-)} = \lim_{t\to\infty} e^{iHt} e^{-iH_a t} P_a. \tag{4.39}$$

The range of $\Omega_a^{(+)}$ is the space of all full states that develop from arrangement channel a, and the range of $\Omega_a^{(-)}$ is the space of all full states that develop into arrangement channel a. Let us call these ranges $\mathcal{R}_a^{(+)}$ and $\mathcal{R}_a^{(-)}$, and their respective orthogonal projections $Q_a^{(+)}$ and $Q_a^{(-)}$. The Möller matrices, $\Omega^{(\pm)}$, map $\mathcal{H}_a$ onto $\mathcal{R}_a^{\pm}$, and from Eqs. (4.37) we find that on $\mathcal{R}_a^{(+)}$ and $\mathcal{R}_a^{(-)}$, respectively,

$$\begin{aligned}\Psi_a(\alpha,t) = \Psi_{\text{in}}(\alpha,t) &= \Omega^{(+)\dagger}\Psi^{(+)}(\alpha,t)\\ = \Psi_{\text{out}}(\alpha,t) &= \Omega^{(-)\dagger}\Psi^{(-)}(\alpha,t).\end{aligned} \tag{4.40}$$

Therefore, because the $\Psi_a(\alpha,t)$ span the space $\mathcal{H}_a$, we find that the Möller matrices, $\Omega_a^{(\pm)}$, are partially isometric from the space $\mathcal{H}_a$, *i.e.*

$$\Omega_a^{(\pm)\dagger}\Omega_a^{(\pm)} = P_a. \tag{4.41}$$

Similarly, the $\Omega_a^{(\pm)\dagger}$ are partially isometric from the ranges $\mathcal{R}_a^{(\pm)}$ of the $\Omega^{(\pm)}$, *i.e.*

$$\Omega_a^{(\pm)}\Omega_a^{(\pm)\dagger} = Q_a^{(\pm)}, \tag{4.42}$$

which defines the $Q_a^{(\pm)}$. The full states developing from or into any arrangement channel are orthogonal to each other as can be seen by direct evaluation of the inner products of asymptotic states. "If the two arrangement channels are different, then there must be at least one particle for which the "overlap" of the two states was negligible in the remote past because it belonged to a different fragment. Hence that inner product must vanish for all times" [33].

A major point of difference with our results from the Rearrangement Model is the statement, "note that the same argument shows that the inner product

$$(\Psi_b(\beta, t), \Psi_a(\alpha, t)) \tag{4.43}$$

approaches zero as $t \to \pm\infty$ (for $a \neq b$). But since $H_a \neq H_b$, it is not independent of t and hence it does not generally vanish for *finite* times" (emphasis added) [33]. In the Rearrangement Model, this is untrue: we showed in Chapter 3 that our states are all orthogonal to each other.

From the Schrödinger equation, one can write

$$H\Omega_a^{(\pm)} = \Omega_a^{(\pm)} H_a, \tag{4.44}$$

which means that Ω intertwines H and H_a. This again is a major difference with the Rearrangement Model, because we showed in Chapter 3 that Ω intertwined H and H_C, where H_C was the comparison Hamiltonian, which had the same spectrum as H; here, H_a does not have the same spectrum as H.

Our channel definitions could also include the single cluster arrangement channel, which is the channel of all the n-particle bound states of H. If we define Λ to be the orthogonal projection onto that subspace, then for all a we have

$$Q_a^{(\pm)} \Lambda = 0. \tag{4.45}$$

Now, every non-bound state must be decomposable into states that arise from, or go into, one of the other arrangements. Therefore, we assume

$$\Lambda + \sum_{a} Q_a^{(\pm)} = 1, \tag{4.46}$$

which is known as asymptotic completeness.

Using Eqs. (4.39), (4.44), and (4.46), we may then define a unitary S-matrix,

$$\Psi^{(+)}(\alpha, t) \quad \underset{t \to \infty}{\rightrightarrows} \quad \Psi_{\text{out}}(t) = \sum_{b} S_{ba} \Psi_a(\alpha, t), \tag{4.47}$$

where

$$S_{ba} = \Omega_b^{(-)\dagger} \Omega_a^{(+)}. \tag{4.48}$$

The discussion in the previous two sections is supposed to be very general. In fact, even though the method described above deals with the non-relativistic region, "the formalism set up is such that, provided there exists a consistent relativistic quantum field theory, the transition to the relativistic domain is relatively simple" [34]. However, we find that even in such a simple model as the Rearrangement Model, these anticipations are not fulfilled. We can, indeed, find a fully orthonormal and complete basis. The formalism above leads to wrong and contradictory results, as will be discussed later in this chapter.

The Almost Local Formalism

In an attempt to avoid splitting up the Hamiltonian, Haag and Ekstein proposed the "almost local" formalism. Following the development of Haag [1], we define a kind of "almost local" product between Heisenberg states. This product refers to a time, t, and we denote it

$$\overset{(t)}{\times} \ . \tag{4.49}$$

We define it by considering two states, $\psi^{(1)}$ and $\psi^{(2)}$, with N_1 and N_2 particles, respectively. If their wave functions at time t are $\psi_t^{(1)}(\vec{x}_1, \ldots, \vec{x}_{N_1})$ and $\psi_t^{(2)}(\vec{x}_2, \ldots, \vec{x}_{N_2})$, respectively, then

$$\psi = \psi^{(1)} \overset{(t)}{\times} \psi^{(2)} \tag{4.50}$$

is a state with $N_1 + N_2$ particles with a Schrödinger wave function at time t given by

$$\begin{aligned} &\psi_t\left(\vec{x}_1, \ldots, \vec{x}_{N_1+N_2}\right) \\ &\quad = C \sum (-1)^P \psi_t^{(1)}\left(\vec{x_{r_1}}, \ldots, \vec{x}_{r_{N_1}}\right) \psi_t^{(2)}\left(\vec{x}_{r_{N_1+1}}, \ldots, \vec{x}_{r_{N_1+N_2}}\right) . \end{aligned} \tag{4.51}$$

The sum in Eq. (4.51) refers to the permutations of coordinates of identical particles. Using the Heisenberg picture, we can work out how ψ depends on time

$$\psi^{(1)} \overset{(t)}{\times} \psi^{(2)} = e^{iHt}\left(e^{-iHt}\psi^{(1)} \overset{(0)}{\times} e^{-iHt}\psi^{(2)}\right), \tag{4.52}$$

and use this information to work out the asymptotic states.

Suppose that there is a Heisenberg state composed of the fragments ζ, η, etc. Then, the product state is

$$\psi^{\zeta} \overset{(t)}{\times} \psi^{\eta} \overset{(t)}{\times} \ldots \ . \tag{4.53}$$

We now use Eq. (4.52), and take the limit as $t \to -\infty$ to get

$$\lim_{t \to -\infty} \psi^{\zeta} \overset{(t)}{\times} \psi^{\eta} \overset{(t)}{\times} \ldots \Rightarrow |\zeta, \eta, \ldots\rangle^{(-)}, \tag{4.54}$$

where $|\zeta, \eta, \ldots\rangle^{(-)}$ means the Heisenberg state initially composed of the fragments ζ, η, etc. We argue that this is true because as $t \to \infty$, the interactions between fragments become negligible because all the wave packets, $\psi_t^{(\zeta)}(\vec{x})$, $\psi_t^{(\eta)}(\vec{x})$, etc., describing the asymptotic center of mass motion of the fragments will have dissolved completely, and the probability of finding two or more fragments within a finite distance of each other in configuration space will be zero. Therefore, asymptotically, Eq. (4.53) will no longer change with time, and thus the strong limit in Eq. (4.54) exists. A similar argument can be made for the asymptotic state when $t \to +\infty$.

The purpose of deriving Eq. (4.54) is that "it is easily applied to more complex collision problems (e.g., quantum field theory with bound states) in which our intuition is not as well developed as in nuclear physics. The only thing needed on the side of the formalism is a suitable definition of the product operation and this is not difficult to find, since it has to satisfy only some qualitative criteria and one therefore has a large amount of freedom in the choice" [35].

The physical situation expressed by the product operation is this: Suppose ϕ_1 and ϕ_2 are two states localized at time t within two volumes V_1 and V_2, respectively, which are far apart. Then

$$\psi_1 \overset{(t)}{\times} \psi_2 \tag{4.55}$$

describes a "doubly localized" state in which the situation within V_1 is described by ψ_1 and within V_2 by ψ_2. This expression has physical meaning only when the two volumes are far apart; when the volumes are close together, the definition is arbitrary.

One now goes on to show the orthonormality and completeness of the asymptotic states. From the physical meaning of the asymptotic product we find, as long as the localization volumes of two states, ψ_1 and ψ_2, are far apart at time t, that

$$\psi_1 \overset{(t)}{\times} \psi_2 = \pm\psi_2 \overset{(t)}{\times} \psi_1, \tag{4.56}$$

the + sign applying for Bose statistics and the − sign for Fermi statistics.

Using this we can define the scalar product

$$\langle\psi_1' \overset{(t)}{\times} \psi_2'|\psi_1 \overset{(t)}{\times} \psi_2\rangle = \langle\psi_1'|\psi_1\rangle\langle\psi_2'|\psi_2\rangle \pm \langle\psi_1'|\psi_2\rangle\langle\psi_2'|\psi_1\rangle. \tag{4.57}$$

Eqs. (4.56) and (4.57) can be checked in wave mechanics. From Eqs. (4.52) and (4.54) we get

$$\begin{aligned}\langle\zeta'\eta'^{(+)}|\zeta\eta^{(+)}\rangle = \lim_{t\to+\infty}\Big\langle \Big[\Big(e^{-iHt}|\zeta'\rangle\Big) \overset{(t)}{\times} \Big(e^{-iHt}|\eta'\rangle\Big)\Big] \\ \Big|\Big[\Big(e^{-iHt}|\zeta\rangle\Big) \overset{(t)}{\times} \Big(e^{-iHt}|\eta\rangle\Big)\Big]\Big\rangle.\end{aligned} \tag{4.58}$$

Because, for large values of $|t|$, only the asymptotic product enters, we now apply Eq. (4.57) to Eq. (4.58) to obtain

$$\langle\zeta'\eta'^{(+)}|\zeta\eta^{(+)}\rangle = \langle\zeta'|\zeta\rangle\langle\eta'|\eta\rangle \pm \langle\zeta'|\eta\rangle\langle\eta'|\zeta\rangle, \tag{4.59}$$

with an analogous result for $|\zeta\eta^{(-)}\rangle$.

To write the orthogonality relations in a general way, we imagine that to every particle type α there corresponds some complete orthonormal basis of states $|\alpha k\rangle$, each k referring to a complete orthonormal system of center of mass wave packets. We abbreviate a configuration $\alpha k, \beta l, \ldots$ by a single Roman letter a. Then, for Bosons, if the state αk appears $n_{\alpha k}$ times in the configuration, we define

$$|a^{(+)}\rangle = \left(n_{\alpha k}! n_{\beta l}! \ldots\right)^{-1/2} |\alpha k, \beta l, \ldots^{(+)}\rangle, \tag{4.60}$$

and therefore,

$$\langle a^{(+)}|b^{(+)}\rangle = \langle a^{(-)}|b^{(-)}\rangle = \delta_{ab}. \tag{4.61}$$

Thus, the states $|a^{(+)}\rangle$ and $|a^{(-)}\rangle$ each form an orthonormal basis system.

Completeness cannot be proved in this formalism because "we must have some general knowledge about the spatial form of $e^{iHt}\phi$ for an arbitrary state ϕ at large times" [36]. Nevertheless, Ruelle extends Haag's work by postulating the completeness of the "in" and "out" states so that "Haag's programme may be carried through rigorously in the framework of the Gårding-Wightman axioms" [37]. In fact, we find in our simple model, with generic form factors, that these asymptotic states are not complete, and are also not normalized.

The S-matrix is isometric since, by definition, S maps each $|a^{(+)}\rangle$ to the corresponding $|a^{(-)}\rangle$, and they are supposed to be orthonormal. The unitarity of S follows from the assumption that the $|a^{(-)}\rangle$ and the $|a^{(+)}\rangle$ sets of basis states are complete.

For the case of field theory, the extension is carried out in the following manner. Suppose $\phi_{\vec{x}}^{\alpha}$ is a state describing one real particle localized at position $\vec{x}$ at time $t = 0$. We want this state to be created by a creation operator $q_{\vec{x}}^{\alpha}$ such that

$$\phi_{\vec{x}}^{\alpha} = q_{\vec{x}}^{\alpha}|0\rangle. \tag{4.62}$$

This equation does not determine $q_{\vec{x}}^{\alpha}$ because it fixes just one column of the infinite matrix q. The natural definition of the product of single particle states,

$$\phi_{\vec{x}_1}^{\alpha} \overset{(t)}{\times} \phi_{\vec{x}_2}^{\alpha} = q_{\vec{x}_1}^{\alpha} q_{\vec{x}_1}^{\alpha}|0\rangle, \tag{4.63}$$

is therefore ambiguous. This ambiguity is removed (essentially by fiat) by defining the creation operators in such a way that only those operators "which are expressible in terms of the basic field quantities of a small space-time environment of the point $\vec{x}$, $t = 0$" [38] are admissible. Such an operator is called "almost local." The rest of the formalism follows the analysis above.

Note that this formalism can only handle particles with Bose and Fermi statistics. Particles with exotic statistics [39] cannot be handled by this type of formalism.

Putting the "generic" formalisms to the test

Potential scattering and the "almost local" formalism have the following protocol for generic scattering systems:

1. They do not use the comparison Hamiltonian
2. The asymptotic states are orthonormal [1,8,30]
3. The completeness of the asymptotic states is postulated [1,30]
4. For the case of potential scattering only, the Möller Matrix (defined as $\lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}$) is isometric but not necessarily unitary [1,30].

We shall take up these points one by one, and put them to the test by comparing them to the results explicitly obtained from our model.

1. It is essential when taking the limits, $\lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}\Psi$, where Ψ is either a wave function or a field operator, that the continuous spectra of H and H_0 coincide. If they did not there would be wild oscillations while taking the limit, and the limit would not exist. It is for this purpose that H_0 is mass-renormalized to H_0'. However, in general, this is still not enough. It is perfectly possible, if there are bound states or unstable particles in the spectrum of H, that no amount of tinkering with H_0 will make its spectrum coincide with H. This can be seen by inspection of Eqs. (3.25) and (3.26). No amount of renormalization of H_0

can give us the discrete M_A state present in H_C, but this may be ignored because M_A is a discrete point eigenvalue. On the other hand, we do have the possibility of a continuous spectrum in H corresponding to the scattering states involving physical B or D particles.

However, unlike H and H_0', H and H_C are guaranteed to be isospectral because H_C is obtained by diagonalizing H. Therefore, it is H_C, and not H_0, that is the proper starting point for any scattering scheme, perturbative* or otherwise. In simple cases such as when stable bound states are not present, or field theory with no bound states or unstable particles, H_C can be identified with the renormalized H_0, as noted in the section *The Möller Matrix and the Comparison Hamiltonian* in Chapter 3.

In fact, even in cases where (formally) no splitting is made, *i.e.* no explicit mention or use is made of an H_0, there is still the implicit use of H_0 because, commonly, asymptotic particles are defined as solutions of free particle equations like the Klein-Gordon equation.

2. Both formalisms assert the orthonormality of the asymptotic states, and the result is supposed to be generic. In Eqs. (3.35), we have obtained the asymptotic states of the Rearrangement Model according to both formalisms. Yet, as we can see from a glance at the Haag-Ruelle asymptotic wave functions, Eqs. (3.35), the asymptotic states do not form an orthonormal set.** This lack

* The method for obtaining the correct spectrum of H by perturbation theory is discussed in the work of Sudarshan, Chiu, and Bhamathi [31].

** This point should not cause confusion. Our full states, namely, Eqs. (2.38), (2.39), (2.40), and (2.41) are, indeed, all orthonormal to each other, as was shown in Chapter 3. As a result, we have orthonormal sets of "in" and "out" states. However, when we calculate of orthonormality stems from a factor of the wave function renormalization constant that appears in each of the asymptotic wave functions. This factor is not a mistake: if it were not present, the interacting states would not be orthonormal.

3. As mentioned earlier, Ruelle extends Haag's work by postulating the completeness of the "in" and "out" states [8]. This is also postulated in simple potential scattering [30]. This postulate is necessary to prove that the S-matrix is unitary. Again, simply by inspection of Eqs. (3.35), we can see that the asymptotic states of the Rearrangement Model, according to these two formalisms, are not complete.§

4. In potential scattering the Möller matrix, Ω, can be defined using the full interacting wave functions so that it is isometric even in the presence of bound states [1]. We see that the Haag-Ruelle asymptotic solutions, Eqs. (3.35), obtained by the use of Ω, are certainly not orthonormal, whereas the original interacting wave functions were; therefore, our Möller matrix is not isometric, *i.e.* it is not norm preserving.

All this points out the importance of the correct normalization of the state vectors, a point already considered by DeWitt [40]. However, his work is not applicable in the case of bound states. The question of the correct description of the asymptotic states was also considered by Van Hove in his papers on the description of "persistent interactions" [4]. However, as

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the Haag-Ruelle type asymptotic states according to either of the formalisms, we find that they are not orthonormal.

§ Again, this point should not cause confusion. Our full states, Eqs. (2.38), (2.39), (2.40), and (2.41) are complete, as was shown in Chapter 3. As a result, our "in" and "out" states form complete sets. However, the set of Haag-Ruelle type asymptotic states calculated according to either of the two formalisms is not complete.

noted in those papers, the formalism developed there also does not deal with cases involving bound states, and does not deal with field theoretic scattering except for a few comments at the end.

In the multichannel case (such as rearrangement collisions), in the "channel Hamiltonian" formalism, the statement is made that the basis states of one group of channels are not orthogonal to the others [9,10,30] because they are eigenstates of different free Hamiltonians. As we can see, in our model, the physical states $C\theta\phi$, $D\phi$, and $B\theta$ are strictly orthogonal to each other. Evidently, this problem arises due to the use of "channel Hamiltonians" in the formalism. It is our belief that the method of splitting up the interaction differently depending on which channel one is considering is fundamentally flawed because "every channel can be distinguished and is observable independently in experiments. This means that these channels should be orthogonal to each other" [41]. One method for ensuring orthonormality is given in [41]; however, this method still suffers from the flaws pointed out above.

It is straightforward to see the problems caused by this lack of orthogonality. We are instructed, in these formalisms, to begin with asymptotic states. Let us first consider the "channel Hamiltonian" formalism. Then, the asymptotic states are the eigenstates of the "channel Hamiltonian" in the sector we are considering. As an example, let us consider

$$|M_B\theta(\omega)\rangle\rangle \rightarrow |M_D\phi(\nu)\rangle\rangle, \tag{4.64}$$

where M_B is the physical B particle and M_D is the physical D particle. We immediately notice, even before we consider any scattering, that the $|M_B\theta(\omega)\rangle\rangle$ state is not orthogonal to the $|M_D\phi(\nu)\rangle\rangle$ state, as can be seen by inspection of Eqs. (2.16), (2.17), (2.18), (2.19), and (2.20). In other words, two experimentally distinct channels are not orthogonal to each other. This will clearly lead to the wrong S-matrix elements because it says that even if there is no scattering, there is a non-zero probability that the $|M_B\theta(\omega)\rangle\rangle$ state will turn into the $|M_D\phi(\nu)\rangle\rangle$ state. We cannot even argue that the two states are "asymptotically orthonormal" [9] because they clearly are not. This can easily be seen by observing that both $|M_B\theta(\omega)\rangle\rangle$ and $|M_D\phi(\nu)\rangle\rangle$ have expansion coefficients in the "bare" $|C\theta(\omega)\phi(\nu)\rangle$ sector. Therefore, as these states are neither orthonormal nor complete, we cannot have an isometric or unitary S-matrix, since orthonormality is necessary for isometry, and completeness for unitarity. However, we have constructed a set of orthonormal (and complete) solutions for our system, a feat that many authors [42] tacitly assume is not possible, and have a perfectly isometric and unitary S-matrix.

Similar problems also arise in the "almost local" formalism. Even if we give that formalism the benefit of the doubt, and say that its asymptotic states are the exact asymptotic states that we constructed (and thus orthogonal but not normalized, instead of neither orthogonal nor normalized), the above problems still occur. Since the asymptotic states are not complete, the S-matrix is not unitary; since they are not orthonormal, the S-matrix is not isometric.

These problems with the S-matrix can be verified by explicit calculation. Since the calculation is tedious, we describe the method, and leave it to the interested reader to verify the results. Our interest is in the scattering of physical states, and so we must start by re-expressing the Hamiltonian, Eqs. (2.3) and (2.4), in terms of the operators which create the *physical* B and D particles. We denote these operators by $\mathcal{B}$ and $\mathcal{D}$, respectively. They are found by inspection of Eqs. (2.16), (2.17), (2.18), (2.19), and (2.20), which are the wave functions for the physical particles. To find the expressions for these operators, we promote the states $|C\phi(\nu)\rangle$, $|C\theta(\omega)\rangle$, $|B\rangle$, and $|D\rangle$ to operators, all acting on the vacuum, and read off the expansions for the operators $\mathcal{B}$ and $\mathcal{D}$. In other words,

$$\mathcal{B}^{\dagger} = \int d\nu \, \rho_B(\nu) C^{\dagger} \phi^{\dagger}(\nu) + Z_B B^{\dagger}, \tag{4.65a}$$

$$\mathcal{D}^{\dagger} = \int d\omega \, \rho_D(\omega) C^{\dagger} \theta^{\dagger}(\omega) + Z_D D^{\dagger}. \tag{4.65b}$$

We re-express the Hamiltonian in terms of these operators, which can be split into various channel Hamiltonians, from which the S-matrix is calculated.

As a physical example, consider the case of a proton bound to a fixed nucleus by a potential V_P, and bombarded by a neutron which interacts with the proton and the nucleus through the potentials V_{PN} and V_N, respectively [41]. The total Hamiltonian of the system is

$$H = K_P + K_N + V_P + V_N + V_{PN}, \tag{4.66}$$

where K_P and K_N are the kinetic energy of the proton and the neutron, respectively. The initial state, denoted by $\Phi_{1,i}$, is given by

$$H_1 \Phi_{1,i} = E_i \Phi_{1,i}, \tag{4.67}$$

where

$$H = H_1 + V_1, \tag{4.68a}$$

$$H_1 = K_P + K_N + V_P, \tag{4.68b}$$

$$V_1 = V_{PN} + V_N. \tag{4.68c}$$

Therefore, the initial state, $\Phi_{1,i}$, is a product of a bound proton, $\phi_P^B(E_i^B)$, and of a free neutron (represented by a plane wave), $u_N(E_i - E_i^B)$, where E_i^B is the binding energy of the proton.

Several possible reactions can occur giving rise to different final products. Let us consider four such reactions.

1. Elastic or inelastic collisions.

   The proton remains bound to the nucleus, and the neutron is free after the collision. Therefore, the Hamiltonian is divided in the same manner as above.

2. Exchange scattering.

   The neutron knocks out the bound proton and becomes bound to the nucleus. The Hamiltonian is then divided as:

$$H = H_2 + V_2, \tag{4.69a}$$

$$H_2 = K_P + K_N + V_N, \tag{4.69b}$$

$$V_2 = V_{PN} + V_P. \tag{4.69c}$$

   Therefore, the final state is

$$H_2 \Phi_{2,f} = E_f \Phi_{2,f}, \tag{4.70a}$$

$$\Phi_{2,f} = u_P \left(E_f - E_f^B\right) \phi_N^B \left(E_f^B\right). \tag{4.70b}$$

3. Ionization.

   The neutron knocks out the bound proton and both are free after the collision. The Hamiltonian is then divided as:

$$H = H_3 + V_3, \tag{4.71a}$$

$$H_3 = K_P + K_N, \tag{4.71b}$$

$$V_3 = V_{PN} + V_P + V_N. \tag{4.71c}$$

   Therefore, the final state is

$$H_3 \Phi_{3,f} = E_f \Phi_{3,f}, \tag{4.72a}$$

$$\Phi_{3,f} = u_P \left(E_f^P\right) u_N \left(E_f - E_f^P\right). \tag{4.72b}$$

4. Pickup.

The proton and the neutron become bound and form a deuteron. The Hamiltonian is then divided as:

$$H = H_4 + V_4, \tag{4.73a}$$

$$H_4 = K_P + K_N + V_{PN}, \tag{4.73b}$$

$$V_4 = V_P + V_N. \tag{4.73c}$$

Therefore, the final state is

$$H_4 \Phi_{4,f} = E_f \Phi_{4,f}, \tag{4.74a}$$

$$\Phi_{4,f} = u_c \left(X, E_f - E_f^B \right) \phi_{PN}^B \left(r, E_f^B \right). \tag{4.74b}$$

Here, X is the center of mass coordinate of the deuteron, and r is the internal coordinate of the deuteron.

The final states given by Eqs. (4.70), (4.72), and (4.74), are eigenstates of different free Hamiltonians. Thus, in general, they are not orthogonal to each other, and the concomitant problems follow.

The reason that these methods do not work properly is that the basis used is one in which bound-state eigenfunctions of the Hamiltonians that bind each fragment are multiplied by plane waves for the fragment motion [30]. In our model, because we have made no breakup, we get the physically reasonable result that the wave functions of the bound states are always orthogonal to the scattering states, and that the basis states of different channels are explicitly orthogonal to each other. We do not have to worry about making the explicit assumption that as the separation between the fragments goes to infinity, the overlap becomes negligible. This assumption may or may not be true, and leads to the problems with "persistent interactions" considered by Van Hove [4].

We compare the results from the two formalisms with those from the Rearrangement Model in Table 4.1.

Property	Potential Scattering	Almost Local formalism	Rearr. Model
Asymp. states norm.?	Yes	Yes	No
Asymp. states orthog.?	Yes	Yes	Yes
Asymp. states compl.?	Yes	Yes	No
Ω isometric?	Yes	N/A	No
S-matrix unitary?	Yes	Yes	Yes
H_C used?	No	No	Yes
Additional property for the multiple-channel case			
Phys. states orthog.?	No	No	Yes

Table 4.1: Comparison of the properties of the Rearrangement Model to various scattering formalisms.

In addition, even when it is not stated explicitly in the literature, it is often assumed that the spectra of the bound states and the scattering (continuum) states do not overlap. As reviewed in Chapter 1, we can construct models in which the spectra of one or more bound states overlap with the continuum. Therefore, this assumption is not necessarily true, and will in general depend upon the details of the model under consideration. We also reviewed in Chapter 1 a demonstration of how two different potentials can lead to the same S-matrix with, in one case, redundant poles unnecessary for completeness, and in the other case, with the same poles being absolutely necessary for completeness. This points out the need for resisting the almost irresistible temptation to identify the poles of the S-matrix with physical bound states of the system.

More importantly, no authors have as yet worried about the evident normalization problem with the asymptotic states because they are always assumed to be normalized. These states are not normalized in the Rearrangement Model, and consequently, assuming orthonormality of the asymptotic states, in general, is very dangerous. In addition, we notice that in this model even though the asymptotic states are not normalized, the interacting states are.

One approach that tries to avoid all these problems, especially in the cases of unstable particles and bound states, is that of analytic continuation [17,23,27,28,43,44] of the state space $\mathcal{H}$ into a generalized vector space $\mathcal{G}$. This has already been done for the case of the Lee model by Parravicini, Gorini, and Sudarshan [24], and by Böhm [45]. For instance, with this method, one can identify resonances and redundant poles, and study the decay of a metastable quantum system. It can also be used for many other things, such as studying the Khalfin observation that the decay of a metastable system with an energy spectrum bounded from below can never be strictly exponential [46]. See the above references for details.

Chapter 5.
Summary and Conclusions

In this work, we constructed a model that allows rearrangement collisions. We explored the spectra and the complete set of orthonormal (ideal) eigenfunctions of this Rearrangement Model in the Rearrangement Sector. Because of the structure of the effective Hamiltonian in this sector, we were able to solve the model exactly. In a similar fashion as for the Cascade Model [17], we find that the spectra can be interpreted as a B particle with energy $M_B < 0$ coupled to a θ particle with energy ω, $0 < \omega < \infty$; a D particle with energy $M_D < 0$ coupled to a θ particle with energy ν, $0 < \nu < \infty$; and a C particle of energy 0 coupled to θ and ϕ particles with energies ω and ν, $0 < \omega, \nu < \infty$. We see that the interacting field theory has a particle interpretation.

Both the B and the D particles suffer mass renormalizations, and these mass renormalizations alter the threshold of the $B\theta$ and $D\phi$ continua, respectively. In Eqs. (2.39b) and (2.40c), we also see the presence of both the mass and wave function renormalizations of the B and D particles in the plane wave parts of their respective wave functions.

We have throughout emphasized the importance of using the comparison Hamiltonian (the diagonalized form of the effective Hamiltonian) because it is isospectral with the full Hamiltonian. Its spectrum differs from that of the free Hamiltonian by the alteration of the $B\theta$ and $D\phi$ continua, and by the addition of a discrete A state. These effects are non-perturbative and, as emphasized in Chapter 4 and in [17], can only be handled by a renormalized perturbation scheme in which H_C, not H_0, is taken as the starting point.

Our results are surprising when compared to what we would expect from conventional scattering theory. We find that while the interacting state vectors are normalized, the asymptotic states are not. Moreover, the asymptotic states are neither orthonormal nor complete. We also find that our physical $C\theta\phi$, $D\phi$, and $B\theta$ states, while being the basis states for different channels, are strictly orthogonal to each other. Further, the Möller matrix in this model is not isometric: it does not preserve the norm of the states. All these results are contrary to the two formalisms of quantum scattering theory we considered. However, these problems are not just special to these two formalisms. Almost every approach to quantum scattering theory makes similar, or the same, assumptions, especially about the isometry of the Möller wave matrix, and the orthonormality and completeness of the asymptotic states. In addition, we reviewed the existence of potentials possessing redundant poles in their S-matrices. These make problematic the identification of poles of the S-matrix with genuine bound states of the system. We also reviewed the existence of potentials possessing bound states "buried" in the continuum. These make problematic the assumption that the energies of bound states are distinct from the continuum.

More generally, we argued that the correct procedure, for any Hamiltonian, H, is to take its complete set of eigenstates, and an associated isospectral comparison Hamiltonian, H_C. The matrix of normalized eigenfunctions of H constitutes the generalized Möller matrix, which is unitary and intertwines H and H_C.

This model is a very simple one. However, even this simple model is enough to show the problems with conventional perturbation theory, and the conventional formulations of scattering theory. It is clearly necessary in the light of this model, and previous work (some of which was reviewed in Chapter 1) on the existence of redundant poles in the scattering amplitude [12,13] and the presence of discrete solutions degenerate in energy with the scattering continuum [16,47], that a fundamental re-examination be made of some of the postulates and assumptions of conventional quantum scattering theory.

Appendix A.
The Lee Model in the Momentum Basis

If we solve the Lee Model in the momentum basis, we find that the results are much more complicated and do not allow for transparent physical interpretation. To show how much more complicated the model becomes in the momentum basis, we solve the model in this basis. Our analysis follows Schweber [48], with slight changes in notation.

Consider the Hamiltonian

$$H = H_0 + H_I, \tag{A.1}$$

with

$$\begin{aligned} H_0 = m_{V_0} \int d\mathbf{p}\, V^\dagger(\mathbf{p}) V(\mathbf{p}) + m_{N_0} \int d\mathbf{p}\, N^\dagger(\mathbf{p}) N(\mathbf{p}) \\ + \int d\mathbf{k}\, \omega_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k}) \end{aligned} \tag{A.2a}$$

and

$$\begin{aligned} H_I = \frac{\lambda_0}{(2\pi)^{3/2}} \int d\mathbf{k}\, \frac{f_1(\mathbf{k}^2)}{\sqrt{2\omega_{\mathbf{k}}}} \int d\mathbf{p} \\ \left\{ V^\dagger(\mathbf{p}) N(\mathbf{p}-\mathbf{k}) a(\mathbf{k}) + N^\dagger(\mathbf{p}-\mathbf{k}) a^\dagger(\mathbf{k}) V(\mathbf{p}) \right\}, \end{aligned} \tag{A.2b}$$

where the field operators V, N, and a are for the V, N and θ quanta, respectively, and obey the usual commutation relations. $\omega_{\mathbf{k}}$ is the energy of the θ particle with mass μ_0 and momentum k; it can be taken to be relativistic or non-relativistic as desired. λ_0 is the coupling constant defining the strength of the interaction, and $f_1(\mathbf{k}^2) = f(\omega_{\mathbf{k}})$ is a cutoff function describing the size of the region over which the interaction takes place.

This Hamiltonian, apart from the total momentum operator $\mathbf{P}$, also has the following constants of motion:

$$Q_1 = \int d\mathbf{p}\, \left(V^\dagger(\mathbf{p})V(\mathbf{p}) + N^\dagger(\mathbf{p})N(\mathbf{p})\right), \tag{A.3a}$$

$$Q_2 = \int d\mathbf{p}\, N^\dagger(\mathbf{p})N(\mathbf{p}) - \int d\mathbf{k}\, a^\dagger(\mathbf{k})a(\mathbf{k}). \tag{A.3b}$$

We can therefore label the eigenstates of H by these quantum numbers.

To solve this model (see Schweber [48] for details) we expand physical states in terms of bare states in the following manner, taking advantage of the fact that only states with the same eigenvalue of Q_1 and Q_2 can occur in the expansion:

$$\begin{aligned} |\ \rangle\rangle = {} & \Phi^{(0,0,0)}|0\rangle + \int d\mathbf{p}\, \Phi^{(1,0,0)}(\mathbf{p})\, V^\dagger(\mathbf{p})|0\rangle + \cdots \\ & + \frac{1}{\sqrt{m!n!l!}} \int d\mathbf{p}_1 \cdots \int d\mathbf{p}_m \int d\mathbf{q}_1 \cdots \int d\mathbf{q}_n \int d\mathbf{k}_1 \cdots \int d\mathbf{k}_l \\ & \Phi^{(m,n,l)}(\mathbf{p}_1, \cdots, \mathbf{p}_m; \mathbf{q}_1, \cdots, \mathbf{q}_n; \mathbf{k}_1, \cdots, \mathbf{k}_l). \end{aligned} \tag{A.4}$$

Assuming that the V particle is stable, and defining

$$F(x) = \frac{{\lambda_0}^2}{(2\pi)^3} \mathrm{P} \int d\mathbf{k}\, \frac{|f(\omega_\mathbf{k})|^2}{2\omega_\mathbf{k}(\omega_\mathbf{k} - x)} \tag{A.5}$$

(analogous to α and β in the Rearrangement Model), the solutions are:

$$\begin{aligned} |V_\mathbf{p}\rangle\rangle = {} & {Z_V}^{1/2} \Big\{ |V_\mathbf{p}\rangle + \frac{\lambda_0}{(2\pi)^{3/2}} \\ & \int d\mathbf{k}\, \frac{f(\omega_\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}\,(m_V - m_N - \omega_\mathbf{k})} |N_{\mathbf{p}-\mathbf{k}}, \theta_\mathbf{k}\rangle \Big\}, \end{aligned} \tag{A.6a}$$

$$\begin{aligned} |N_\mathbf{q}, \theta_\mathbf{k}\rangle\rangle = {} & |N_\mathbf{q}, \theta_\mathbf{k}\rangle \\ & + \frac{\lambda_0}{(2\pi)^{3/2}} \frac{f(\omega_\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}} \frac{1}{m_N + \omega_\mathbf{k} - H + i\epsilon} V^\dagger(\mathbf{q} + \mathbf{k})\,|0\rangle, \end{aligned} \tag{A.6b}$$

where one finds

$$m_V = m_{V_0} - F(m_V - m_N), \tag{A.7a}$$

$${Z_V}^{-1} = 1 + \left(\frac{dF}{dx}\right)_{x = m_V - m_N}, \tag{A.7b}$$

and

$$\frac{1}{m_N+\omega_{\mathbf{k}}-H+i\epsilon}|V_{\mathbf{q}+\mathbf{k}}\rangle=\frac{1}{G_+(\omega_{\mathbf{k}})}\frac{1}{m_N+\omega_{\mathbf{k}}-m_V}$$
$$\left\{|V_{\mathbf{q}+\mathbf{k}}\rangle+\frac{\lambda_0}{(2\pi)^{3/2}}\int d\mathbf{k}'\,\frac{f(\omega_{\mathbf{k}'})}{\sqrt{2\omega_{\mathbf{k}'}}(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}+i\epsilon)}|N_{\mathbf{q}+\mathbf{k}-\mathbf{k}'},\theta_{\mathbf{k}'}\rangle\right\},\quad (A.8a)$$

with

$$G_+(\omega_{\mathbf{k}})=1+\frac{\lambda_0{}^2}{(2\pi)^3}\int d\mathbf{k}'\,\frac{|f(\omega_{\mathbf{k}'})|^2}{(m_V-m_N-\omega_{\mathbf{k}'})\,2\omega_{\mathbf{k}'}}\frac{1}{(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}+i\epsilon)}.\quad (A.8b)$$

Looking at these solutions, and comparing them with the Lee Model solutions that we used for the lower sector of the Rearrangement Model, Eqs. (2.14), we can see that the latter are much simpler and more easily understood. It therefore behooves us to absorb the coupling constants, phase space factors, and other miscellaneous objects in H into one "form factor" function, and work in the energy basis where things are much simpler and the physical interpretation is much more transparent.

Finally, let us look at (Schweber's result for) the amplitude for $N\theta \to N\theta$ scattering. Defining $S_{ba}=\delta(a-b)-2\pi i\delta(E_a-E_b)R_{ba}$, we find

$$\langle N_{\mathbf{q}'},\theta_{\mathbf{k}'}|R|N_{\mathbf{q}},\theta_{\mathbf{k}}\rangle=\frac{Z_V\lambda_0{}^2}{(2\pi)^3}\frac{|f(\omega_{\mathbf{k}})|^2}{2\omega_{\mathbf{k}}}\delta^{(3)}\left(\mathbf{k}+\mathbf{q}-\mathbf{k}'-\mathbf{q}'\right)\frac{1}{m_N+\omega_{\mathbf{k}}-m_V}$$
$$\left[1+\frac{Z_V\lambda_0{}^2}{(2\pi)^3}\int d\mathbf{k}''\frac{|f(\omega'')|^2(m_V-m_N-\omega_{\mathbf{k}})}{2\omega''(m_V-m_N-\omega'')(\omega_{\mathbf{k}}-\omega''+i\epsilon)}\right]^{-1}.\quad (A.9)$$

Notice that this result is independent of the direction of the momentum. Therefore, there is only s-wave scattering, just as in the Cascade model of Chiu, Sudarshan, and Bhamathi [17], and in our Rearrangement Model.

Appendix B.
Some Useful Formulae

The following formulae are very useful for the calculations in the main text. By our definitions in Chapter 2, we have the following ranges for our variables:

$$0 \leq \lambda \leq \infty, \tag{B.1}$$

$$0 \leq \mu \leq \infty, \tag{B.2}$$

$$0 \leq n \leq \infty, \tag{B.3}$$

with E being free to run over all values.

We then have the easily proved identities

$$|g(\lambda)|^2 = \frac{1}{2\pi i}\left[\beta(\lambda) - \beta^*(\lambda)\right], \tag{B.4}$$

$$|f(\mu)|^2 = \frac{1}{2\pi i}\left[\alpha(\mu) - \alpha^*(\mu)\right], \tag{B.5}$$

$$\frac{|g(\lambda)|^2}{|\beta(\lambda)|^2} = -\frac{1}{2\pi i}\left[\frac{1}{\beta(\lambda)} - \frac{1}{\beta^*(\lambda)}\right], \tag{B.6}$$

$$\frac{|f(\mu)|^2}{|\alpha(\mu)|^2} = -\frac{1}{2\pi i}\left[\frac{1}{\alpha(\mu)} - \frac{1}{\alpha^*(\mu)}\right], \tag{B.7}$$

$$\frac{|g(E-\lambda)|^2}{|\beta(E-\lambda)|^2} = -\frac{1}{2\pi i}\left[\frac{1}{\beta(E-\lambda)} - \frac{1}{\beta^*(E-\lambda)}\right] - |Z_B|^2\delta(E-\lambda-M_B), \tag{B.8}$$

$$\frac{|f(E-\mu)|^2}{|\alpha(E-\mu)|^2} = -\frac{1}{2\pi i}\left[\frac{1}{\alpha(E-\mu)} - \frac{1}{\alpha^*(E-\mu)}\right] - |Z_D|^2\delta(E-\mu-M_D). \tag{B.9}$$

The equations (B.8) and (B.9) follow because $E-\lambda$ and $E-\mu$ can be less than zero, and thus pick up singularities at $M_B < 0$ and $M_D < 0$, respectively.

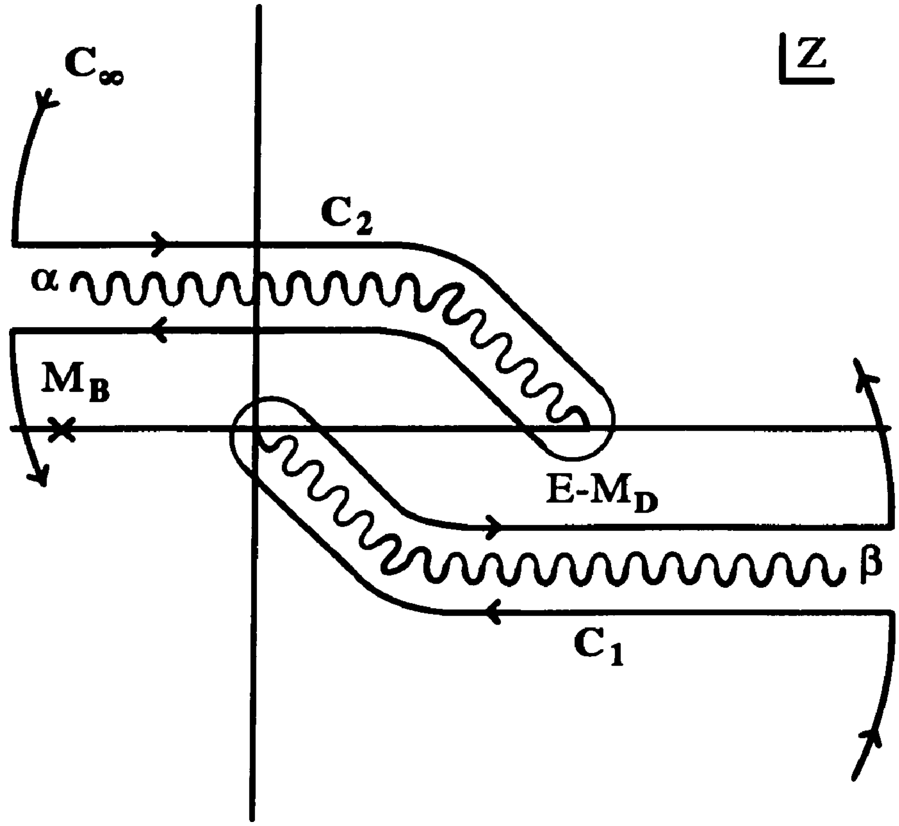

Figure B.1: Contour for Eq. (B.10a)

On the other hand λ and μ are always greater than or equal to zero, and so cannot pick up any singularities.

Another useful identity is

$$\gamma(E) = \int d\lambda \, \frac{|g(\lambda)|^2}{|\beta(\lambda)|^2} \frac{1}{\alpha(E-\lambda)} + \frac{|Z_B|^2}{\alpha(E-M_B)} \qquad (B.10a)$$

$$= \int d\mu \, \frac{|f(\mu)|^2}{|\alpha(\mu)|^2} \frac{1}{\beta(E-\mu)} + \frac{|Z_D|^2}{\beta(E-M_D)}. \qquad (B.10b)$$

We can easily show this by means of the contours in Fig. B.1 and Fig. B.2. If we convert the integral in Eq. (B.10a) into a contour integral by using

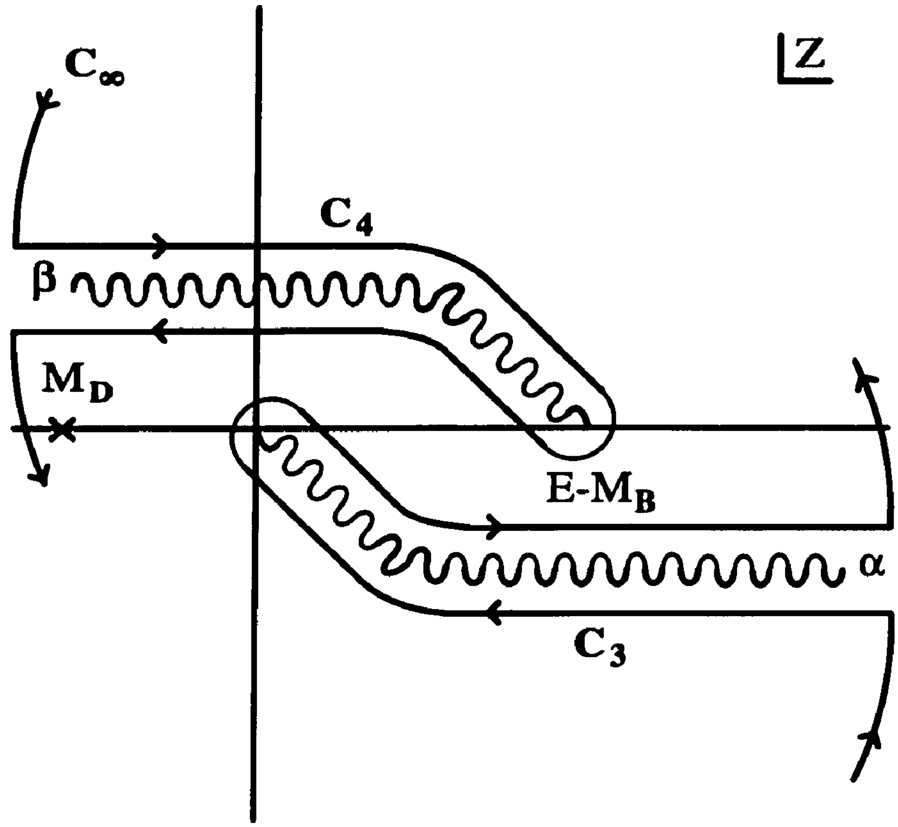

Figure B.2: Contour for Eq. (B.10b)

Eq. (B.6), we get

$$\left(-\frac{1}{2\pi i}\right)\int_{C_1} dz\,\frac{1}{\beta(z)\,\alpha(E-z)}+\frac{|Z_B|^2}{\alpha(E-M_B)}, \qquad (B.11)$$

with the contour shown in Fig. B.1. Then, we make a change of variables from z to $E-z$ to get

$$\left(-\frac{1}{2\pi i}\right)(-1)\int_{C_4} dz\,\frac{1}{\alpha(z)\,\beta(E-z)}+\frac{|Z_B|^2}{\alpha(E-M_B)}, \qquad (B.12)$$

with the contour shown in Fig. B.2. Now we deform the contour C_4 and write it as the contour C_3 plus the circle at infinity, while picking up the contributions from the residues of the integrand. Note that the circle at infinity gives no result, so we have

$$\left(-\frac{1}{2\pi i}\right)(-1)(-1)\int_{C_3} dz\,\frac{1}{\alpha(z)\beta(E-z)} + \left(-\frac{1}{2\pi i}\right)(-1)(2\pi i)\frac{|Z_D|^2}{\alpha(E-M_D)}$$
$$+\left(-\frac{1}{2\pi i}\right)(-1)(-1)(2\pi i)\frac{|Z_B|^2}{\alpha(E-M_B)} + \frac{|Z_B|^2}{\alpha(E-M_B)}. \qquad (B.13)$$

The Z_B terms cancel, and the first two terms are Eq. (B.10b), by definition. Therefore, Eq. (B.10a) is equal to Eq. (B.10b), and the identity is established.

We can similarly show that

$$\int d\lambda\,\frac{|g(\lambda)|^2}{|\beta(\lambda)|^2}\frac{1}{\alpha(E-\lambda)}\frac{1}{(\lambda-\nu+i\epsilon)}$$
$$=\int d\mu\,\frac{|f(\mu)|^2}{|\alpha(\mu)|^2}\frac{1}{\beta(E-\mu)}\frac{1}{(E-\mu-\nu+i\epsilon)} - \frac{1}{\alpha(E-\nu)\beta(\nu)}$$
$$+\frac{|Z_D|^2}{\beta(E-M_D)(E-M_D-\nu+i\epsilon)} - \frac{|Z_B|^2}{\alpha(E-M_B)(E-M_B-\omega+i\epsilon)}. \qquad (B.14)$$

Using Eqs. (B.7), (B.8), and (B.10) we can get another useful formula

$$\int dn\,\frac{|f(n)|^2|g(E-n)|^2}{|\alpha(n)|^2|\beta(E-n)|^2}$$
$$=\frac{\gamma(E)-\gamma^*(E)}{(-2\pi i)} - |Z_D|^2\frac{|g(E-M_D)|^2}{|\beta(E-M_D)|^2} - |Z_B|^2\frac{|f(E-M_B)|^2}{|\alpha(E-M_B)|^2}. \qquad (B.15)$$

Finally, in Fig. B.3, we display the branch cuts and poles of $\frac{1}{\gamma(z)}$ which are used in showing the completeness of our solution set.

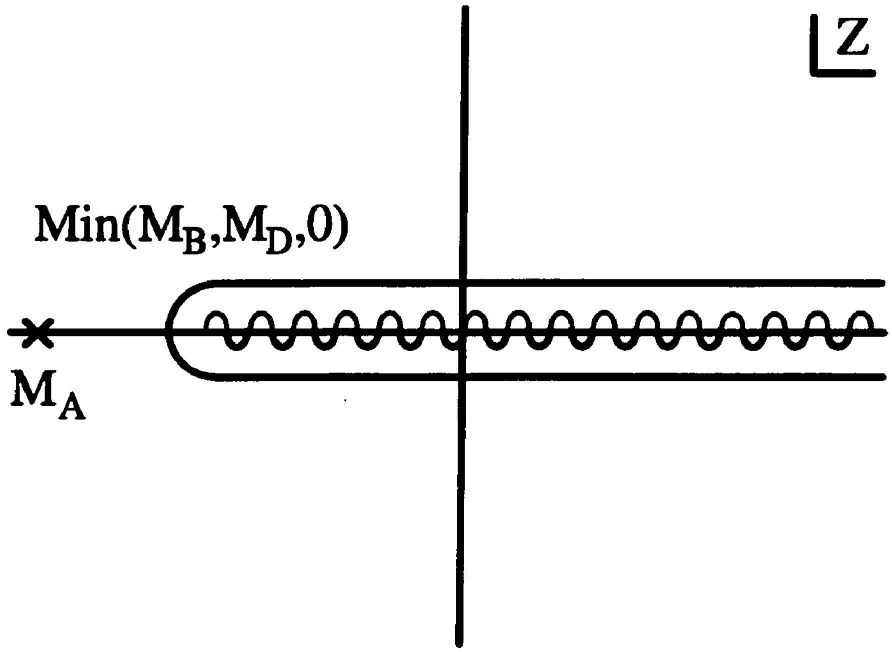

Figure B.3: Contour for the function $\frac{1}{\gamma(z)}$.

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